Properties of the curve $x^y = y^x$

Russ Gordon

Whitman College

June 28, 2019

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We restrict ourselves to the set of positive real numbers.

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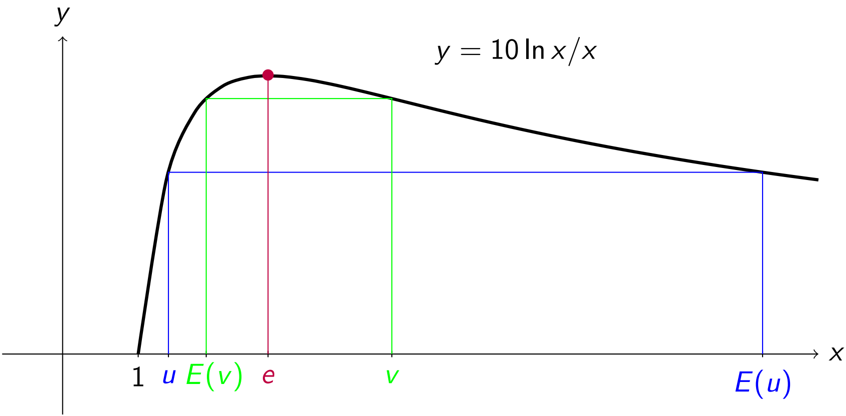
It follows that $\dfrac{\ln x}{x} = \dfrac{\ln E(x)}{E(x)}$.

Hence, solutions to $x^y = y^x$ correspond to horizontal chords on the graph of $\beta(x) = \dfrac{\ln x}{x}$.



It is clear that $\frac{\ln u}{u} = \frac{\ln E(u)}{E(u)}$, $\frac{\ln v}{v} = \frac{\ln E(v)}{E(v)}$, and $E(e) = e$.

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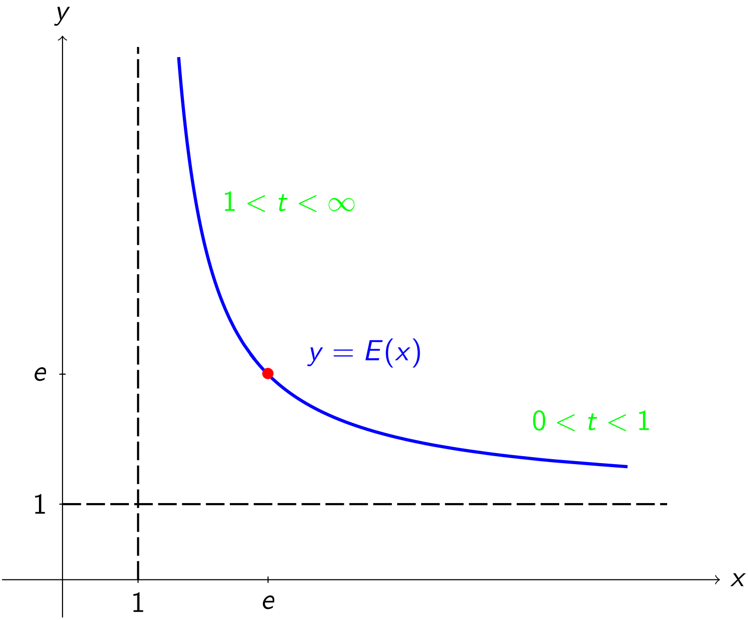
Note that $y(t) = t\,x(t)$ and $y(t) = (x(t))^t$.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Rational solutions $x < y$ if and only if s is a positive integer.

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$$t = g(f(t)) = g(\ln x) \quad \text{and thus} \quad E(x) = y = t\,x = x\,g(\ln x).$$

The inequality

$$\frac{2}{t+1} \leq f(t) = \frac{\ln t}{t-1} \leq \frac{1}{\sqrt{t}}$$

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$$\frac{2}{t+1} = 1 - \frac{1}{2}(t-1) + \frac{1}{4}(t-1)^2 + \cdots$$

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The equivalence

$$\frac{2}{t+1} < \frac{\ln t}{t-1} \quad \Leftrightarrow \quad 2 < \ln x^{t+1},$$

reveals that

$$E(x) = y = x^t = \frac{x^{t+1}}{x} > \frac{e^2}{x}.$$

$$\frac{e^2}{x} = e - (x - e) + \frac{1}{e}(x - e)^2 + \cdots$$

$$E(x) = e - (x - e) + \frac{5}{3e}(x - e)^2 + \cdots$$

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The three functions are very close near e, but there is a vast discrepancy as $x \to 1^+$ and $x \to \infty$.

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Given the range of the function $e^{f(t)}$, it is sufficient to prove that $E'(e^{f(t)}) < 0$ and $E''(e^{f(t)}) > 0$ for all $t > 0$.

$$E(x) = x\, g(\ln x);$$

$$E(e^{f(t)}) = t\, e^{f(t)};$$

$$E'(e^{f(t)})f'(t) = t\, f'(t) + 1; \;\left(\text{want } t\, f'(t) + 1 > 0\right)$$

$$E''(e^{f(t)})e^{f(t)}f'(t) = 1 - \frac{f''(t)}{(f'(t))^2}. \;\left(\text{want } f''(t) > (f'(t))^2\right)$$

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$$(t-1)f''(t)+2f'(t) = -\frac{1}{t^2} \quad \Leftrightarrow \quad t^2(t-1)^2f''(t) = 1-3t+2t^2f(t)$$

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This gets very messy. (But it does work.)

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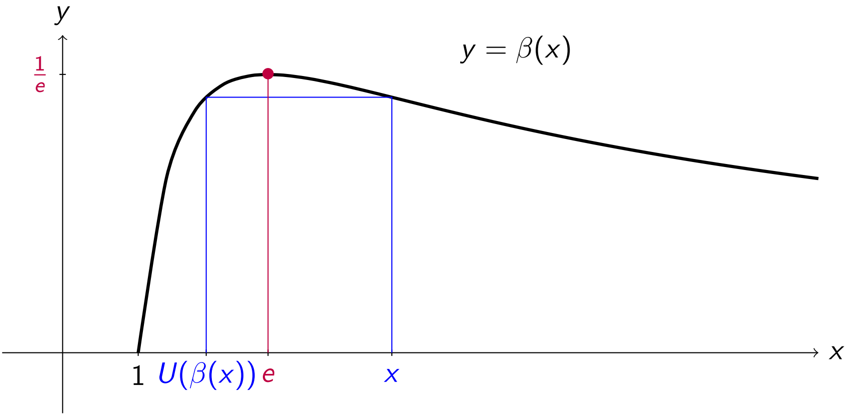
This also works but, as you may envision, it is extremely tedious.



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This gets us close, but alas, we need the interval $[e, \infty)$.

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For this approach, we need to use an abstract function and, more importantly, we still come up short.

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Thanks for listening.