$\mathbb{R}^n$ is the set of all n-tuples of real numbers

$\mathcal{C}([a,b])$ is the set of all continuous functions defined on $[a,b]$

$\mathcal{P}_n$ is the set of all polynomials of degree less than or equal to n

$\mathcal{P}$ is the set of all polynomials

c_0 is the set of all sequences $\{a_k\}$ for which $\{a_k\}$ converges to 0

ℓ_1 is the set of all sequences $\{a_k\}$ for which the series $\sum_{k=1}^{\infty} |a_k|$ converges

ℓ_2 is the set of all sequences $\{a_k\}$ for which the series $\sum_{k=1}^{\infty} |a_k|^2$ converges

ℓ_∞ is the set of all sequences $\{a_k\}$ for which $\{a_k\}$ is bounded

A metric space (X,d) consists of a set X and a real-valued function d defined on $X \times X$ that satisfies the following four properties.

1. $d(x,y) \geq 0$ for all $x, y \in X$.
2. $d(x,y) = 0$ if and only if $x = y$.
3. $d(x,y) = d(y,x)$ for all $x, y \in X$.
4. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

The function d, which gives the distance between two points in X, is known as a metric.

A normed linear space X is a vector space X with a norm defined on it. A norm on a vector space X is a real-valued function defined on X with the following four properties:

1. $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in X$.
2. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
3. $\|c\,\mathbf{x}\| = |c|\|\mathbf{x}\|$ for all $\mathbf{x} \in X$ and $c \in \mathbb{R}$.
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in X$.

A norm on X defines a metric on X via the definition $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

An inner product space X is a vector space X with an inner product defined on it. An inner product on X is a real-valued function defined on $X \times X$ with the following five properties:

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in X$.
2. $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
3. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}, \mathbf{y} \in X$.
4. $\langle c\,\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in X$ and $c \in \mathbb{R}$.
5. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$.

An inner product on X defines a norm on X via the definition $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Every normed linear space is a metric space.

Every inner product space is a normed linear space and thus a metric space.

One metric on the set ℓ_∞ is given by $d(\{a_k\},\{b_k\}) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{2^k}$.

Another metric on the set ℓ_∞ is given by $d(\{a_k\},\{b_k\}) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{2^k(1+|a_k - b_k|)}$.

A norm on the set $\mathcal{C}([a,b])$ is given by $\|x(t)\| = \max\{|x(t)| : t \in [a,b]\}$.

Another norm on the set $\mathcal{C}([a,b])$ is given by $\|x(t)\| = \sqrt{\int_a^b (x(t))^2\,dt}$.

A norm on the set ℓ_1 is given by $\|\{a_k\}\| = \sum_{k=1}^{\infty} |a_k|$.

An inner product on the set $\mathcal{C}([a,b])$ is given by $\langle x(t), y(t)\rangle = \int_a^b x(t)y(t)\,dt$.

This inner product generates the norm $\|x(t)\| = \sqrt{\int_a^b (x(t))^2\,dt}$.

An inner product on the set ℓ_2 is given by $\langle \{a_k\},\{b_k\}\rangle = \sum_{k=1}^{\infty} a_k b_k$.

This inner product generates the norm $\|\{a_k\}\| = \sqrt{\sum_{k=1}^{\infty} a_k^2}$.

Let $\{x_n\}$ be a sequence in a metric space (X,d). The sequence $\{x_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

A metric space (X,d) is a complete metric space if every Cauchy sequence in (X,d) converges to a point in X.

A Banach space is a normed linear space that is complete under the metric defined by its norm.

A Hilbert space is an inner product space that is complete under the metric defined by it inner product.

The metric $d(\{a_k\},\{b_k\}) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{2^k(1+|a_k - b_k|)}$ on ℓ_∞ does not come from a norm.

The set $\mathcal{C}([a,b])$ with the norm $\|x(t)\| = \max\{|x(t)| : t \in [a,b]\}$ is a Banach space.

The set $\mathcal{C}([a,b])$ with the norm $\|x(t)\| = \sqrt{\int_a^b (x(t))^2\,dt}$ is not a Banach space.

The set ℓ_2 with the inner product $\langle \{a_k\},\{b_k\}\rangle = \sum_{k=1}^{\infty} a_k b_k$ is a Hilbert space.

Cauchy-Schwarz Inequality:

In any inner product space, we have $|\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ for all $\mathbf{x}$ and $\mathbf{y}$.

Assume that $\mathbf{y} \ne \mathbf{0}$ and let $c = \dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$. We then have

$$\begin{aligned}
0 \le \|\mathbf{x} - c\,\mathbf{y}\|^2 &= \langle \mathbf{x} - c\,\mathbf{y}, \mathbf{x} - c\,\mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, c\,\mathbf{y} \rangle - \langle c\,\mathbf{y}, \mathbf{x} \rangle + \langle c\,\mathbf{y}, c\,\mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - c\,\langle \mathbf{x}, \mathbf{y} \rangle - c\big(\langle \mathbf{x}, \mathbf{y} \rangle - c\,\langle \mathbf{y}, \mathbf{y} \rangle\big) \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - c\,\langle \mathbf{x}, \mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\langle \mathbf{y}, \mathbf{y} \rangle} \\
&= \|\mathbf{x}\|^2 - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^2} \\
&= \frac{\|\mathbf{x}\|^2\|\mathbf{y}\|^2 - \langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^2}
\end{aligned}$$

and thus

$$\langle \mathbf{x}, \mathbf{y} \rangle^2 \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \qquad \Leftrightarrow \qquad |\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

Triangle Inequality:

In any inner product space, we have $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}$ and $\mathbf{y}$.

For vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\
&\le \langle \mathbf{x}, \mathbf{x} \rangle + 2|\langle \mathbf{x}, \mathbf{y} \rangle| + \langle \mathbf{y}, \mathbf{y} \rangle \\
&= \|\mathbf{x}\|^2 + 2|\langle \mathbf{x}, \mathbf{y} \rangle| + \|\mathbf{y}\|^2 \\
&\le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2
\end{aligned}$$

and thus $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

In the inner product space $\mathcal{C}([0,6])$, find an orthogonal basis for span $\{1, t, t^2\}$.

In the inner product space $\mathcal{C}([1,2])$, find an orthogonal basis for span $\{t, \ln t\}$.

$$\langle t, t \rangle = \int_1^2 t^2 \, dt = \frac{t^3}{3} \Big|_1^2 = \frac{7}{3}$$

$$\langle t, \ln t \rangle = \int_1^2 t \ln t \, dt = \left(\frac{1}{2} \, t^2 \ln t - \frac{1}{4} \, t^2 \right) \Big|_1^2 = 2 \ln 2 - \frac{3}{4}$$

$$\ln t - \frac{\langle t, \ln t \rangle}{\langle t, t \rangle} \, t = \ln t - \frac{2 \ln 2 - \frac{3}{4}}{\frac{7}{3}} \, t = \ln t + \frac{3}{7} \left(\frac{3}{4} - 2 \ln 2 \right) t = \ln t + \left(\frac{9}{28} - \frac{6}{7} \ln 2 \right) t$$

An orthogonal basis is thus $\{t, 28 \ln t + (9 - 24 \ln 2) \, t\}$.