Let $\{c_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let a be a real number. A power series centered at a is an expression of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots.$$

The constants c_k are known as the coefficients of the power series and the number a is called the center of the power series. A power series looks and behaves like an infinite degree polynomial.

Let $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, where the power series has radius of convergence $\rho > 0$. Then f is infinitely differentiable on $(a-\rho, a+\rho)$ and $f^{(k)}(a) = k!\, c_k$ for each $k \geq 0$. ■

Turning things around, we can start with an infinitely differentiable function f, find its derivatives, then write the resulting power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \qquad \text{(For many, but not all functions, this equals } f(x).)$$

This series is called the Taylor series of f centered at a or, in the frequently occurring case in which $a = 0$, the **Maclaurin series** of f. Three common Maclaurin series are

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k = 1 + x + \frac{1}{2}\, x^2 + \frac{1}{6}\, x^3 + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = x - \frac{1}{6}\, x^3 + \frac{1}{120}\, x^5 - \frac{1}{5040}\, x^7 + \cdots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} = 1 - \frac{1}{2}\, x^2 + \frac{1}{24}\, x^4 - \frac{1}{720}\, x^6 + \cdots$$

An inner product on the set $\mathcal{C}([a,b])$ is given by $\langle x(t), y(t)\rangle = \int_a^b x(t)y(t)\, dt$.

The set $\{1, \cos t, \sin t, \cos(2t), \sin(2t), \cos(3t), \sin(3t), \ldots\}$ is an orthogonal basis for $\mathcal{C}([-\pi,\pi])$.

A Fourier series is a series of the form

$$\sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) = a_0 + a_1 \cos t + b_1 \sin t + a_2 \cos(2t) + b_2 \sin(2t) + \cdots.$$

Fourier series have much stranger behavior than power series. For example, the function f defined by $f(t) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n t)$ is continuous on $\mathbb{R}$ but nowhere differentiable on $\mathbb{R}$.

Since we have an orthogonal basis, it is easy to write down a formula for the coefficients ($n \geq 1$):

$$a_n = \frac{\int_{-\pi}^{\pi} f(t)\cos(nt)\, dt}{\int_{-\pi}^{\pi} \cos^2(nt)\, dt} \qquad \text{and} \qquad b_n = \frac{\int_{-\pi}^{\pi} f(t)\sin(nt)\, dt}{\int_{-\pi}^{\pi} \sin^2(nt)\, dt}.$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$
$$\sin(x-y) = \sin x \cos y - \sin y \cos x$$
$$2\sin x \cos y = \sin(x+y) + \sin(x-y)$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$2\cos x \cos y = \cos(x+y) + \cos(x-y)$$
$$2\sin x \sin y = \cos(x-y) - \cos(x+y)$$

$$\sin 2x = 2\sin x \cos x \qquad 2\sin^2 x = 1 - \cos 2x$$
$$\cos 2x = \cos^2 x - \sin^2 x \qquad 2\cos^2 x = 1 + \cos 2x$$

$$\int_{-\pi}^{\pi} \sin(nt)\,dt = -\frac{1}{n}\cos(nt)\Big|_{-\pi}^{\pi} = -\frac{1}{n}\big((-1)^n - (-1)^n\big) = 0$$
$$\int_{-\pi}^{\pi} \cos(nt)\,dt = \frac{1}{n}\sin(nt)\Big|_{-\pi}^{\pi} = \frac{1}{n}\big(0-0\big) = 0$$
$$2\int_{-\pi}^{\pi} \sin(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi}\big(\sin((m+n)t) + \sin((m-n)t)\big)\,dt = 0$$
$$2\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \int_{-\pi}^{\pi}\big(\cos((m-n)t) - \cos((m+n)t)\big)\,dt = 0$$
$$2\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi}\big(\cos((m+n)t) + \cos((m-n)t)\big)\,dt = 0$$
$$\int_{-\pi}^{\pi} \sin^2(nt)\,dt = \frac{1}{2}\int_{-\pi}^{\pi}\big(1 - \cos(2nt)\big)\,dt = \pi$$
$$\int_{-\pi}^{\pi} \cos^2(nt)\,dt = \frac{1}{2}\int_{-\pi}^{\pi}\big(1 + \cos(2nt)\big)\,dt = \pi$$

For a continuous function f on $[-\pi, \pi]$, it then follows that

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) = a_0 + a_1\cos t + b_1 \sin t + a_2\cos(2t) + b_2\sin(2t) + \cdots,$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt.$$

A function f is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
A function f is odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
The product of two even functions is an even function.
The product of two odd functions is an even function.
The product of an even function and an odd function is an odd function.
If f is an odd function and $a > 0$, then $\int_{-a}^{a} f(x)\,dx = 0$.
If f is an even function and $a > 0$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

Find the Fourier series for the function $f(t) = t$ on the interval $[-\pi, \pi]$.

Noting that f is an odd function and using integration by parts, we find that

$$\int_{-\pi}^{\pi} t\cos(nt)\,dt = 0 \quad \text{for all } n \geq 0;$$

$$\int_{-\pi}^{\pi} t\sin(nt)\,dt = 2\int_{0}^{\pi} t\sin(nt)\,dt = 2\Big(-\frac{t}{n}\cos(nt)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nt)\,dt\Big) = -\frac{2(-1)^n\pi}{n}$$

for all $n \geq 1$. It follows that

$$t = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nt) = 2\sin t - \sin(2t) + \tfrac{2}{3}\sin(3t) - \tfrac{1}{2}\sin(4t) + \cdots.$$

This equation is valid for all t in the interval $(-\pi, \pi)$. In particular, when $t = \frac{1}{2}\pi$, we find that

$$\begin{aligned}\frac{\pi}{4} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\pi/2)\\ &= \frac{1}{1}\sin(\pi/2) - 0 + \frac{1}{3}\sin(3\pi/2) - 0 + \frac{1}{5}\sin(5\pi/2) - 0 + \frac{1}{7}\sin(7\pi/2) + \cdots\\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}.\end{aligned}$$

Find the Fourier series for the function $f(t) = t^3$ on the interval $[-\pi, \pi]$.

Noting that f is an odd function and using integration by parts, we find that

$$\begin{aligned}\int_{-\pi}^{\pi} t^3\cos(nt)\,dt &= 0 \quad \text{for all } n \geq 0;\\ \int_{-\pi}^{\pi} t^3\sin(nt)\,dt &= 2\int_0^{\pi} t^3\sin(nt)\,dt\\ &= 2\Big(-\frac{t^3}{n}\cos(nt)\Big|_0^{\pi} + \frac{3}{n}\int_0^{\pi} t^2\cos(nt)\,dt\Big)\\ &= -\frac{2(-1)^n\pi^3}{n} + \frac{6}{n}\cdot\frac{2(-1)^n\pi}{n^2}.\end{aligned}$$

It follows that

$$t^3 = \sum_{n=1}^{\infty}(-1)^{n+1}\Big(\frac{2\pi^2}{n} - \frac{12}{n^3}\Big)\sin(nt) = (2\pi^2 - 12)\sin t - (\pi^2 - \tfrac{3}{2})\sin(2t) + (\tfrac{2}{3}\pi^2 - \tfrac{4}{9})\sin(3t) - \cdots.$$

This equation is valid for all t in the interval $(-\pi, \pi)$. In particular, when $t = \frac{1}{2}\pi$, we find that

$$\begin{aligned}\frac{\pi^3}{8} &= \pi^2\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(n\pi/2) - \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{n^3}\sin(n\pi/2)\\ &= \pi^2\cdot\frac{\pi}{2} - 12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin(n\pi/2)\end{aligned}$$

and thus

$$\frac{\pi^3}{32} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin(n\pi/2) = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}.$$

Find the Fourier series for the function $f(t) = \begin{cases} 1, & \text{if } |t| \le \pi/2; \\ 0, & \text{if } |t| > \pi/2; \end{cases}$ on the interval $[-\pi, \pi]$.

Noting that f is an even function, we find that $\displaystyle\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = 0$ for all $n \ge 1$ and

$$\int_{-\pi}^{\pi} f(t)\,dt = \pi;$$

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = 2\int_{0}^{\pi/2} \cos(nt)\,dt &= \frac{2}{n}\,\sin(nt)\Big|_0^{\pi/2} \\ &= \frac{2\sin(n\pi/2)}{n} = \begin{cases} 2/n, & \text{if } n = 1, 5, 9, 13, \ldots; \\ -2/n, & \text{if } n = 3, 7, 11, 15, \ldots; \end{cases} \\ &\text{which we can write as } \frac{2(-1)^{n+1}}{2n-1}.\end{aligned}$$

It follows that

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(2n-1)\pi}\cos((2n-1)t) = \frac{1}{2} + \frac{2}{\pi}\Big(\cos t - \tfrac{1}{3}\cos(3t) + \tfrac{1}{5}\cos(5t) - \tfrac{1}{7}\cos(7t) + \cdots\Big).$$

This equation is valid for all t in the interval $(-\pi, \pi)$ for which f is continuous. When $t = 0$, using an earlier result, we find that series for $f(0)$ does indeed equal 1. Let's see what happens then $t = \pi/3$. We hope to have

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}\cos(\tfrac{1}{3}(2n-1)\pi).$$

Writing out the series, we have

$$\frac{\cos(\frac{1}{3}\pi)}{1} - \frac{\cos(\pi)}{3} + \frac{\cos(\frac{5}{3}\pi)}{5} - \frac{\cos(\frac{7}{3}\pi)}{7} + \frac{\cos(3\pi)}{9} - \frac{\cos(\frac{11}{3}\pi)}{11} + \frac{\cos(\frac{13}{3}\pi)}{13} - \frac{\cos(5\pi)}{15} + \frac{\cos(\frac{17}{3}\pi)}{17} + \cdots,$$

then using the values of cosine at $\pi/3$, π, and $5\pi/3$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}\cos(\tfrac{1}{3}(2n-1)\pi) &= \frac{1}{2} + \frac{1}{3} + \frac{1}{10} - \frac{1}{14} - \frac{1}{9} - \frac{1}{22} + \frac{1}{26} + \frac{1}{15} + \frac{1}{34} - \cdots \\ &= \frac{1}{2}\Big(1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \cdots\Big) + \frac{1}{3}\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big) \\ &\quad + \frac{1}{2}\Big(-\frac{1}{3} + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} - \cdots\Big) - \frac{1}{2}\Big(-\frac{1}{3} + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} - \cdots\Big) \\ &= \frac{1}{2}\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big) + \Big(\frac{1}{3} + \frac{1}{6}\Big)\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big) \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.\end{aligned}$$

The sum of this series is indeed $\pi/4$ from our previous work.