Name: due October 14 by 3:00 pm Pacific

Math 126 **Homework Assignment 17** **Fall 2020**

1. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for each positive integer n. (Imitate the proof given for Theorem 3.1 in the textbook.)

We will use the PMI. Let S be the set of all positive integers n for which $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. Since $1^3 = \frac{1^2 \cdot 2^2}{4}$, we see that $1 \in S$. Now suppose that $k \in S$ for some positive integer k. This means $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$. We then have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right) \\
&= \frac{(k+1)^2}{4}(k+2)^2 \\
&= \frac{(k+1)^2\left((k+1)+1\right)^2}{4}.
\end{aligned}$$

This shows that $k+1 \in S$. Hence we have proved that $k \in S$ implies $k+1 \in S$. By the PMI, the set S equals $\mathbb{Z}^+$. Therefore $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all positive integers n.

notes: you should check some numbers first

$1 = \frac{1^2 \cdot 2^2}{4}$

$1 + 8 = \frac{2^2 \cdot 3^2}{4}$

$1 + 8 + 27 = \frac{3^2 \cdot 4^2}{4}$

also the factoring is guided by the fact that we know what we hope to get

2. Prove that for each positive integer n, the integer $7^n + 2^{2n+1}$ is divisible by 3. (Imitate the proof of Theorem 3.2 in the textbook.)

$n = 0$ $\quad 7^0 + 2^1 = 3 = 3 \cdot 1$ (not necessary)

$n = 1$ $\quad 7^1 + 2^3 = 15 = 3 \cdot 5$

$n = 2$ $\quad 7^2 + 2^5 = 81 = 3 \cdot 27$

$n = 3$ $\quad 7^3 + 2^7 = 471 = 3 \cdot 157$

$$\begin{array}{r} 343 \\ 128 \\ \hline 471 \end{array}$$

all divisible by 3

We will use the PMI. Since $7 + 2^3 = 15 = 3 \cdot 5$, the statement is true when $n = 1$. Now suppose that $7^k + 2^{2k+1}$ is divisible by 3 for some positive integer k. By definition, there exists an integer j such that $7^k + 2^{2k+1} = 3j$. Then

$$\begin{aligned} 7^{k+1} + 2^{2k+3} &= 7 \cdot 7^k + 4 \cdot 2^{2k+1} \\ &= 7\left(3j - 2^{2k+1}\right) + 4 \cdot 2^{2k+1} \\ &= 7(3j) - 3 \cdot 2^{2k+1} \\ &= 3\left(7j - 2^{2k+1}\right), \end{aligned}$$

which shows that $7^{k+1} + 2^{2(k+1)+1}$ is divisible by 3. By the PMI, the integer $7^n + 2^{2n+1}$ is divisible by 3 for every positive integer n.