Name: due November 11 by 3:00 pm Pacific

Math 126 **Homework Assignment 27** **Fall 2020**

1. Use a geometric series to find a power series expression centered at 1 for the function $f(x) = \dfrac{2}{7-4x}$ and determine the interval of convergence for the resulting series. (Note that finding the interval should be very easy and require minimal effort.)

$$f(x) = \frac{2}{7-4x} = \frac{2}{3-4(x-1)} = \frac{2/3}{1-\frac{4}{3}(x-1)} \qquad a = 2/3 \qquad r = \frac{4}{3}(x-1)$$

$$f(x) = \sum_{k=0}^{\infty} \frac{2}{3}\left(\frac{4}{3}(x-1)\right)^k = \sum_{k=0}^{\infty} \frac{2\cdot 4^k}{3\cdot 3^k}(x-1)^k$$

$$= \sum_{k=0}^{\infty} \frac{2^{2k+1}}{3^{k+1}}(x-1)^k$$

The series converges when $\left|\frac{4}{3}(x-1)\right| < 1 \iff |x-1| < \frac{3}{4}$.

The interval of convergence is $\left(\frac{1}{4}, \frac{7}{4}\right)$.

2. Use results in Section 3.11 (as well as some basic ideas) to find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2+k+1}{4^k}$.

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}, \quad \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}, \quad \sum_{k=1}^{\infty} k^2x^k = \frac{x+x^2}{(1-x)^3}.$$

$$\sum_{k=1}^{\infty} \frac{3k^2+k+1}{4^k} = 3\sum_{k=1}^{\infty} k^2\left(\frac{1}{4}\right)^k + \sum_{k=1}^{\infty} k\left(\frac{1}{4}\right)^k + \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k$$

$$= 3\cdot\frac{\frac{1}{4}+\frac{1}{16}}{(3/4)^3} + \frac{1/4}{(3/4)^2} + \frac{1/4}{3/4}$$

$$= \frac{3}{27}(16+4) + \frac{1}{9}\cdot 4 + \frac{1}{3}$$

$$= \frac{20}{9} + \frac{4}{9} + \frac{3}{9}$$

$$= 3.$$

The sum of the series is 3.

3. Starting with the solution to problem 2 in Section 3.11 (it can be found in the back of the book) and taking derivatives, find a formula for the sum of the series $\sum_{k=1}^{\infty} k^3 x^k$. Use your formula to find $\sum_{k=1}^{\infty} \frac{k^3}{3^k}$.

$$\frac{x+x^2}{(1-x)^3} = \sum_{k=1}^{\infty} k^2 x^k$$

differentiate both sides then multiply by x

$$x \frac{d}{dx} \sum_{k=1}^{\infty} k^2 x^k = x \sum_{k=1}^{\infty} k^3 x^{k-1} = \sum_{k=1}^{\infty} k^3 x^k$$

$$x \frac{d}{dx} \frac{x+x^2}{(1-x)^3} = x\left(\frac{(1-x)^3(1+2x) - (x+x^2)\,3(1-x)^2(-1)}{(1-x)^6}\right)$$

$$= x\left(\frac{(1-x)(1+2x) + 3(x+x^2)}{(1-x)^4}\right)$$

$(1-x)^2$ cancels

$$= x\left(\frac{1+x-2x^2+3x+3x^2}{(1-x)^4}\right)$$

$$= \frac{x+4x^2+x^3}{(1-x)^4}$$

Therefore $\sum_{k=1}^{\infty} k^3 x^k = \frac{x+4x^2+x^3}{(1-x)^4}$ for $|x|<1$.

In particular, we find that

$$\sum_{k=1}^{\infty} \frac{k^3}{3^k} = \sum_{k=1}^{\infty} k^3\left(\frac{1}{3}\right)^k = \frac{\frac{1}{3}+\frac{4}{9}+\frac{1}{27}}{(2/3)^4} = \frac{22}{27}\cdot\frac{81}{16}$$

$$= \frac{22}{16}\cdot 3 = \frac{33}{8}.$$