Name: due November 13 by 3:00 pm Pacific

Math 126 **Homework Assignment 28** **Fall 2020**

1. Use the Maclaurin series for $\sin x$ to find the Maclaurin series for the function $\frac{\sin x - x}{x^3}$.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Here are the three relevant Maclaurin series.

$$\sin x - x = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\frac{\sin x - x}{x^3} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k-2} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+3)!} x^{2k}$$

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\frac{\sin x - x}{x^3} = -\frac{1}{6} + \frac{1}{120}x^2 - \frac{1}{7!}x^4 + \cdots$$

2. Use the Maclaurin series for $\sin x$ to find the Maclaurin series for the function $\int_0^x \frac{\sin t}{t}\, dt$.

$$\frac{\sin t}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k}$$

$$\int_0^x \frac{\sin t}{t}\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^x t^{2k}\, dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+1)!} x^{2k+1}$$

This represents the requested Maclaurin series.

$$\frac{\sin t}{t} = 1 - \frac{1}{6}t^2 + \frac{1}{120}t^4 - \frac{1}{7!}t^6 + \cdots$$

$$\int_0^x \frac{\sin t}{t}\, dt = x - \frac{1}{18}x^3 + \frac{1}{600}x^5 - \frac{1}{7\cdot 7!}x^7 + \cdots$$

3. Use known Maclaurin series to represent the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k+1}$ as a more familiar function.

The $k!$ term indicates an e^x type function.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k+1} = x \sum_{k=0}^{\infty} \frac{1}{k!} (-x^2)^k$$

$$= x e^{-x^2}.$$

4. For the function $f(x) = xe^{x^3}$, find $f^{(61)}(0)$. (First find the Maclaurin series for f, then look again at Theorem 3.23.)

The Maclaurin series for f is

$$f(x) = x e^{x^3} = x \sum_{k=0}^{\infty} \frac{1}{k!} (x^3)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^{3k+1}.$$

We know that $f^{(61)}(0) = 61!\, c_{61}$. To find c_{61}, we let $k = 20$. [We need the coefficient on the x^{61} term.] It follows that $c_{61} = \frac{1}{20!}$ and thus $f^{(61)}(0) = \frac{61!}{20!}$.