A Brief Summary of Math 125

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{v \to x} \frac{f(v) - f(x)}{v - x} \quad \text{or (equivalently)} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for each value of x in the domain of f for which the limit exists.

The number $f'(c)$ represents the **slope** of the graph $y = f(x)$ at the point $(c, f(c))$. It also represents the **rate of change** of y with respect to x when x is near c. This interpretation of the derivative leads to applications that involve **related rates**, that is, related quantities that change with time.

An equation for the **tangent line** to the curve $y = f(x)$ when $x = c$ is $y - f(c) = f'(c)(x - c)$. The **normal line** to the curve is the line that is perpendicular to the tangent line.

Using the definition of the derivative, it is possible to establish the following **derivative formulas**. Other useful techniques involve **implicit differentiation** and **logarithmic differentiation**.

$$\frac{d}{dx}\, x^r = r x^{r-1},\ r \neq 0 \qquad \frac{d}{dx}\,\sin x = \cos x \qquad \frac{d}{dx}\,\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\,\ln|x| = \frac{1}{x} \qquad \frac{d}{dx}\,\cos x = -\sin x \qquad \frac{d}{dx}\,\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\, e^x = e^x \qquad \frac{d}{dx}\,\tan x = \sec^2 x \qquad \frac{d}{dx}\,\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\,\log_a|x| = \frac{1}{(\ln a)x},\ a > 0 \qquad \frac{d}{dx}\,\cot x = -\csc^2 x \qquad \frac{d}{dx}\,\operatorname{arccot} x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\, a^x = (\ln a)\, a^x,\ a > 0 \qquad \frac{d}{dx}\,\sec x = \sec x \tan x \qquad \frac{d}{dx}\,\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\,\csc x = -\csc x \cot x \qquad \frac{d}{dx}\,\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

product rule: $\dfrac{d}{dx}\big(F(x)G(x)\big) = F(x)G'(x) + G(x)F'(x)$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{F(x)}{G(x)}\right) = \dfrac{G(x)F'(x) - F(x)G'(x)}{(G(x))^2}$ **chain rule**: $\dfrac{d}{dx}\, F\big(G(x)\big) = F'\big(G(x)\big)\, G'(x)$

Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

This theorem states that there is a point c for which the instantaneous rate of change of f at c ($f'(c)$) equals the average (mean) rate of change of f on $[a, b]$ ($(f(b) - f(a))/(b - a)$). Graphically, it states that the tangent line at some point is parallel to the secant line through the endpoints. This is an extremely important theorem and it can be used to prove the following three facts.

1. If f' is positive (negative) on an interval I, then f is **increasing** (**decreasing**) on I. This fact makes it possible to use f' to determine the values of x for which f has a **relative maximum value** or a **relative minimum value**. The first step is to find the **critical points** of f: points x in the domain of f for which either $f'(x) = 0$ or $f'(x)$ does not exist. Then the **First Derivative Test** can be used to determine the nature of the critical point.
2. If f'' is positive (negative) on an interval I, then f is **concave up** (**concave down**) on I. An **inflection point** occurs where the graph changes concavity. Possible inflection points occur when $f''(x) = 0$, but it is necessary to check that the concavity actually changes at such points (consider $f(x) = x^4$).

3. If $f' = g'$ on an interval I, then there is a constant C such that $g(x) = f(x) + C$ for all x in I.

Fact (1) is used to solve max/min word problems since it helps locate the highs and lows of a function. Of course, the challenge with word problems is determining the correct function to use; this takes practice and experience. Facts (1) and (2) together make it possible to generate relatively accurate graphs of a function without the aid of a calculator. They help to identify portions of the graph where interesting things occur. Speaking of graphing, the following two definitions are also relevant when sketching graphs.

A function f has a **vertical asymptote** $x = c$ if either $\lim_{x \to c^-} |f(x)| = \infty$ or $\lim_{x \to c^+} |f(x)| = \infty$.

A function f has a **horizontal asymptote** $y = d$ if either $\lim_{x \to \infty} f(x) = d$ or $\lim_{x \to -\infty} f(x) = d$.

Fact (3) is crucial for integration; once we find one antiderivative of a function, we know that all of the other antiderivatives just differ by a constant (this is the basis for the $+C$ term for antiderivatives).

A function f is **continuous** at a number c if $\lim_{x \to c} f(x) = f(c)$. This fact guarantees that the graph of f does not have a break at c. An important theorem states: If f is differentiable at c, then f is continuous at c. You should know how to prove this simple fact. However, the converse is false; the function $f(x) = |x|$ is continuous at 0 but not differentiable at 0. Graphically, such functions have a sharp corner or a vertical tangent at these "bad" points. The following results are two important theorems for continuous functions.

Intermediate Value Theorem: If f is continuous on a closed interval $[a, b]$ and v is any number between $f(a)$ and $f(b)$, then there is a number c in (a, b) such that $f(c) = v$.

Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then there exist numbers c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all x in $[a, b]$. (The number $f(c)$ is the minimum value of f on $[a, b]$ and the number $f(d)$ is the maximum value of f on $[a, b]$.)

The IVT guarantees that continuous functions do not "skip" values while the EVT states that continuous functions have max and min values (but note that the interval must be of the form $[a, b]$). These values either occur at the endpoints of the interval or at interior critical points.

Although it is not necessary to know, the mathematical definition of the crucial concept of limit is

Definition of limit: Let f be defined on some open interval containing the point c, except possibly at c. Then $\lim_{x \to c} f(x) = L$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x that satisfy $0 < |x - c| < \delta$. (Note that the value of f at c (even if it is defined) is irrelevant.)

Various algebraic techniques (factoring, expanding, finding a common denominator, multiplying by the conjugate) can be used to evaluate limits. You should be familiar with these AND be able to use them to compute derivatives from the definition. The following rule is sometimes useful for computing limits of the form $0/0$ or ∞/∞; these are known as **indeterminate forms**. The suitable conditions mentioned in the hypotheses involve continuity and differentiability conditions that will always be met by the functions we encounter. (Note clearly that you **cannot** use this rule to compute derivatives from the definition.)

L'Hôpital's Rule: Under suitable conditions on the functions f and g, if either $\lim_{x \to *} f(x) = 0 = \lim_{x \to *} g(x)$ or $\lim_{x \to *} f(x) = \infty = \lim_{x \to *} g(x)$, then $\lim_{x \to *} \frac{f(x)}{g(x)} = \lim_{x \to *} \frac{f'(x)}{g'(x)}$, assuming that the latter limit exists. (The limits here can be of any type; $x \to c$, $x \to c^+$, $x \to c^-$, $x \to \infty$, $x \to -\infty$.)

For the record, there are other indeterminate forms such as $\infty - \infty$, $0 \cdot \infty$, ∞^0, and 0^0 for which L'Hôpital's Rule does not apply. You should be familiar with techniques that are helpful in these cases.