1. If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

For $g(x) = \dfrac{x^2+3x}{x-1}$, we have $g'(x) = \dfrac{(x-1)(2x+3)-(x^2+3x)}{(x-1)^2}$

or $g'(x) = \dfrac{x^2-2x-3}{(x-1)^2}$.

Since $g(-1) = \dfrac{1-3}{-2} = 1$ and $g(0) = 0$, we need a number $c \in (-1,0)$ such that $\dfrac{c^2-2c-3}{(c-1)^2} = \dfrac{0-1}{0-(-1)} = -1$.

Algebra yields
$$\begin{aligned} c^2-2c-3 &= -c^2+2c-1 \\ 2c^2-4c-2 &= 0 \\ c^2-2c-1 &= 0 \\ (c-1)^2 &= 2 \qquad \text{(completing the square)} \\ c &= 1 \pm \sqrt{2} \end{aligned}$$

Since c must be negative, we find that $c = 1-\sqrt{2}$.

2. We start by finding $f''(x)$ and solving $f''(x) = 0$.

$f(x) = 4e^{-x^2/10}$

$f'(x) = 4e^{-x^2/10}\left(-\frac{x}{5}\right) = -\frac{4}{5}xe^{-x^2/10}$

$f''(x) = -\frac{4}{5}\left[xe^{-x^2/10}\left(-\frac{x}{5}\right) + e^{-x^2/10}\right] = \frac{4}{25}e^{-x^2/10}(x^2-5)$

$f''(x) = 0$ when $x = \pm\sqrt{5}$

Since $e^{-x^2/10}$ is always positive, we see that f'' changes sign at each of these points so inflection points do occur at $\pm\sqrt{5}$. Note that $f(\pm\sqrt{5}) = 4e^{-1/2}$. The inflection points are $(-\sqrt{5}, 4/\sqrt{e})$ and $(\sqrt{5}, 4/\sqrt{e})$.

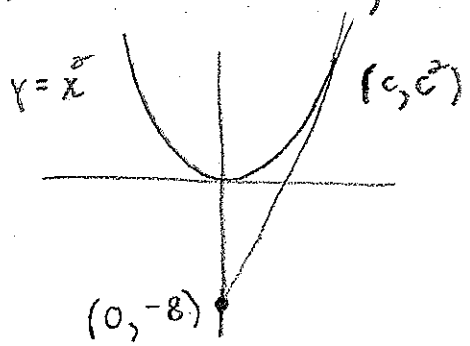
3. We are given $y = x\sqrt{3x+1}$ so $y = 2$ when $x = 1$. To find the slope of the tangent line, we need the derivative. Using the product rule,

$$\frac{dy}{dx} = x \cdot \frac{3}{2\sqrt{3x+1}} + \sqrt{3x+1}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4} + 2 = \frac{11}{4}$$

The tangent line goes through $(1,2)$ with slope $11/4$ so its equation is $y - 2 = \frac{11}{4}(x-1)$.

4. Note that $(0,-8)$ is not on the graph of $y = x^2$.



Referring to the graph, we need to find c so that $2c = \frac{c^2+8}{c-0}$; the slope of the tangent line is the same as the slope of the line to $(0,-8)$.

Solving for c yields $2c^2 = c^2 + 8$ or $c = \pm 2\sqrt{2}$. Using the positive value gives us the point $(2\sqrt{2}, 8)$ and the tangent line at this point is given by $y - 8 = 4\sqrt{2}\,(x - 2\sqrt{2})$ or $y = 4\sqrt{2}\,x - 8$. Note that this line goes through $(0,-8)$.

5. Consider the function $f(x) = 16x^4 + ax$, where a is any positive constant. It follows that

$$\lim_{x \to \infty} \frac{3x^2 + 4x - 600}{\sqrt{f(x)}} = \lim_{x \to \infty} \frac{3x^2 + 4x - 600}{\sqrt{16x^4 + ax}}$$

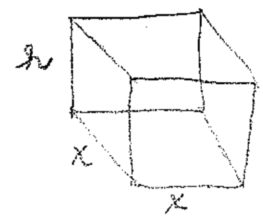
$$= \lim_{x \to \infty} \frac{3x^2 + 4x - 600}{\sqrt{16x^4 + ax}} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x \to \infty} \frac{3 + \frac{4}{x} - \frac{600}{x^2}}{\sqrt{16 + a/x^3}}$$

$$= \frac{3}{4}.$$

Since there are many choices for a, the function f is not unique.

6. We sketch a picture and introduce some letters to represent the quantities, using feet as our units.



$V = x^2 h$, volume of box

$1 \cdot x^2$, cost for the top; $4 \cdot x^2$, cost for the base;

$2 \cdot 4hx$, cost for the sides

We want to maximize V given $5x^2 + 8hx = 80$ or $h = \frac{80 - 5x^2}{8x}$. Since x and h must be positive, we must maximize $V(x) = x^2 \cdot \frac{80 - 5x^2}{8x} = \frac{5}{8}(16x - x^3)$ for $0 < x < 4$. Note that $V'(x) = 0 \Rightarrow 16 = 3x^2 \Rightarrow x = \frac{4}{\sqrt{3}}$ and this value clearly corresponds to a maximum value on the interval $(0, 4)$. The volume of the largest box is $V\left(\frac{4}{\sqrt{3}}\right) = \frac{5}{8} \cdot \frac{4}{\sqrt{3}}\left(16 - \frac{16}{3}\right) = \frac{5}{2\sqrt{3}} \cdot \frac{2}{3} \cdot 16 = \frac{80}{3\sqrt{3}}$ cubic feet.

(The dimensions are $\frac{4}{\sqrt{3}} \times \frac{4}{\sqrt{3}} \times \frac{5}{\sqrt{3}}$ for this optimal box.)

7. Using the usual $s(t)$, $v(t)$, $a(t)$ notation, we are given that $a(t) = -6t$, $v(0) = 120$, and $s(0) = 0$. It follows that

$$a(t) = -6t$$
$$v(t) = -3t^2 + 120$$
$$s(t) = -t^3 + 120t$$

The particle stops when $v(t) = 0$ or $t = \sqrt{40}$. Since $s(\sqrt{40}) = -40\sqrt{40} + 120\sqrt{40} = 80\sqrt{40} = 160\sqrt{10}$, we find that the particle travels $160\sqrt{10}$ feet before coming to a stop.

8. By the Mean Value Theorem, we seek a point $c \in (a,b)$ so that

$$-\frac{1}{c^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a},$$

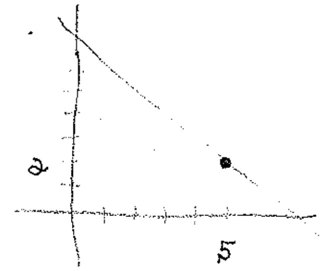
since $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$. Some algebra yields

$$\frac{1}{c^2} = \frac{\frac{1}{a} - \frac{1}{b}}{b-a} = \frac{\frac{b-a}{ab}}{b-a} = \frac{1}{ab}$$

and it follows that $c = \sqrt{ab}$.

(Note that c is the geometric mean of a and b.)

9. Let m be the slope, where $-\infty < m < 0$.



The equation of the line is

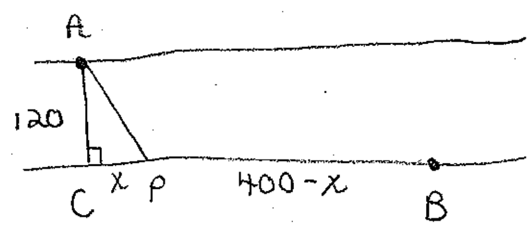
$y - 2 = m(x-5)$

x-intercept $5 - \frac{2}{m}$

y-intercept $2 - 5m$

The area $A(m)$ of the triangle is $A(m) = \frac{1}{2}\left(5 - \frac{2}{m}\right)(2-5m)$.
Since $A(m) = \frac{1}{2}\left(20 - 25m - \frac{4}{m}\right)$, we have $A'(m) = \frac{1}{2}\left(\frac{4}{m^2} - 25\right)$.
We see that $A'(m) = 0$ when $m = -\frac{2}{5}$. Since $A'(m) < 0$ for $-\infty < m < -\frac{2}{5}$ and $A'(m) > 0$ for $-\frac{2}{5} < m < 0$, it follows that A has a minimum when $m = -\frac{2}{5}$. Therefore, the line $y - 2 = -\frac{2}{5}(x-5)$ or $2x + 5y = 20$ cuts off the triangle with the least area.

10.

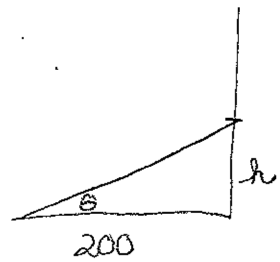


Let x be the number of feet past C where the cable crosses the bog. Then $PB = 400 - x$ and $AP = \sqrt{x^2 + 120^2}$. The cost of the cable is $C(x) = 50(400-x) + 100\sqrt{x^2+120^2}$, where $0 \le x \le 400$.

$$C'(x) = 50\left(-1 + \frac{2x}{\sqrt{x^2+120^2}}\right) \text{ so } C'(x) = 0 \Rightarrow \begin{array}{l} 4x^2 = x^2 + 120^2 \\ x = \frac{120}{\sqrt{3}} = 40\sqrt{3}. \end{array}$$

Since $C'(x)$ goes from negative to positive, we have a minimum when $x = 40\sqrt{3}$. The cable should go to the point P that is $40\sqrt{3}$ feet from C toward B, then go from P to B the remaining distance.

11. Referring to the figure, we see that $\tan\theta = \frac{h}{200}$ and



thus $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{200} \cdot \frac{dh}{dt}$

and $\frac{d\theta}{dt} = \frac{1}{200}\cos^2\theta \cdot \frac{dh}{dt}$.

At the given instant $\frac{dh}{dt} = 30$ and $\cos\theta = \frac{200}{\sqrt{200^2+80^2}}$.

Therefore $\left.\frac{d\theta}{dt}\right|_{h=80} = \frac{1}{200} \cdot \frac{200^2}{200^2+80^2} \cdot 30 = \frac{200 \cdot 30}{40^2(5^2+2^2)}$

$= \frac{15}{4 \cdot 29} = \frac{15}{116}$

The angle of observation is increasing at $\frac{15}{116}$ rad/min.

12. Rather than solve for y, we use implicit differentiation.

$$2x + 3x\frac{dy}{dx} + 3y - 4y\frac{dy}{dx} - 1 + 5\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1-2x-3y}{3x-4y+5}$$

It follows that $\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{1-4-3}{6-4+5} = -\frac{6}{7}$.

13. We will use L'Hopital's Rule since we have a $\frac{0}{0}$ form

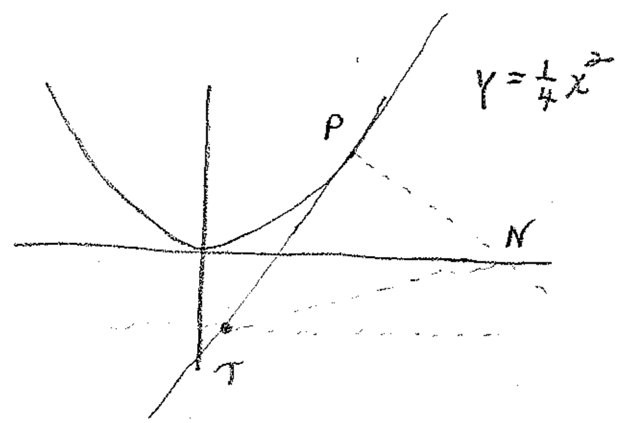
$$\lim_{x\to 0} \frac{\ln(1+x) - \sin x}{1-\cos^2 x} = \lim_{x\to 0} \frac{\frac{1}{1+x} - \cos x}{2\cos x \sin x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x} - \cos x}{\sin 2x}$$

$$= \lim_{x\to 0} \frac{\frac{-1}{(1+x)^2} + \sin x}{2\cos 2x}$$

$$= -\frac{1}{2}$$

14. At the point $(6,9)$, the slope of the tangent line is $\frac{1}{2}x\Big|_{x=6} = 3$. Hence,



Tangent line $\quad y - 9 = 3(x-6)$

$y = -3 \Rightarrow \quad -12 = 3(x-6) \Rightarrow x = 2 \qquad T(2,-3)$

normal line $\quad y - 9 = -\frac{1}{3}(x-6)$

$y = 0 \Rightarrow \quad -9 = -\frac{1}{3}(x-6) \Rightarrow x = 33 \qquad N(33,0)$

We have a right triangle so $A = \frac{1}{2}\cdot \overline{PT} \cdot \overline{PN}$.

Using the distance formula, we have

$$A = \frac{1}{2}\sqrt{4^2+12^2}\,\sqrt{27^2+9^2} = \frac{1}{2}\cdot 4\sqrt{10}\cdot 9\sqrt{10} = 180.$$

The area of the triangle is 180 square units.

15. Given $y = \ln(x^2 + 2x - 2)$, we see that $y = 0$ when $x = 1$ and $\frac{dy}{dx} = \frac{2x+2}{x^2+2x-2}$ so $\left.\frac{dy}{dx}\right|_{x=1} = 4$. An equation for the tangent line is $y = 4x - 4$.

16. Using one version of the definition, we have

$$f'(x) = \lim_{v \to x} \frac{\frac{1}{v} - \frac{1}{x}}{v - x} = \lim_{v \to x} \frac{1}{v-x} \cdot \frac{x-v}{vx}$$

$$= \lim_{v \to x} \frac{-1}{vx} = -\frac{1}{x^2}.$$

17. The max/min values occur at endpoints or interior critical points.

$g(x) = 30x - x^3$

$g'(x) = 30 - 3x^2$

$g'(x) = 0 \Rightarrow x = \pm\sqrt{10}$

x	$g(x)$	
1	29	m
$\sqrt{10}$	$20\sqrt{10}$	M
4	56	

The minimum value of g is 29 and the maximum value of g is $20\sqrt{10}$.

18. With the usual notation:

$a(t) = -32$

$v(t) = -32t + 24$

$s(t) = -16t^2 + 24t + 160$

$s(t) = 0 \Rightarrow 2t^2 - 3t - 20 = 0 \Rightarrow (2t+5)(t-4) = 0$

$v(4) = -128 + 24 = -104$

The rock hits the ground after 4 sec with a downward speed of 104 ft/sec.

19. We first find $\frac{dy}{dx}$ using implicit differentiation.

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x\frac{dy}{dx} + 6y \Rightarrow \frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2}$$

The tangent line is horizontal when $y = x^2/2$ (making $\frac{dy}{dx} = 0$) so

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x \cdot \frac{x^2}{2} \Rightarrow 1 + \frac{1}{8}x^3 = 3 \quad (\text{since } x \neq 0)$$

$$x = \sqrt[3]{16} = 2\sqrt[3]{2}$$

$$y = \frac{4\sqrt[3]{4}}{2} = 2\sqrt[3]{4}$$

The tangent line is horizontal at $(2\sqrt[3]{2}, 2\sqrt[3]{4})$.

20. For continuity, we need $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^+} g(x)$

or $4 - a = 8b + 3$. We know $g'(x) = \begin{cases} 2x, & x < 2 \\ 3bx^2, & x > 2 \end{cases}$

so $4 = 12b$ is needed for $g'(2)$ to be defined.

It follows that $b = \frac{1}{3}$ and $a = 1 - 8b = -\frac{5}{3}$.

21. Using the definition, we find that

$$\frac{d}{dx}\sqrt{f(x)} = \lim_{h \to 0} \frac{\sqrt{f(x+h)} - \sqrt{f(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h\left(\sqrt{f(x+h)} + \sqrt{f(x)}\right)}$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot \frac{1}{\sqrt{f(x+h)} + \sqrt{f(x)}}\right)$$

$$= f'(x) \cdot \frac{1}{2\sqrt{f(x)}} = \frac{f'(x)}{2\sqrt{f(x)}}.$$

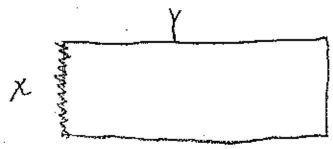
Note that we used the fact that f is continuous when we took the limit.

22. The square of the distance from $(x, \frac{2}{x^2})$ to $(0,0)$ is $S(x) = x^2 + \frac{4}{x^4}$. To minimize this, note that

$$S'(x) = 2x - \frac{16}{x^5} = \frac{2(x^6-8)}{x^5},$$

which shows that $x = \pm\sqrt{2}$ are the critical points. These values clearly correspond to minimum values of $S(x)$. The minimum distance is thus $\sqrt{S(\sqrt{2})} = \sqrt{3}$.

23. Let x and y be the lengths of the sides (in meters), with x being the more expensive side.



We know $xy = 40$ and we want to minimize the cost $k(x+2y) + 2kx$, where k is some constant. Since $y = \frac{40}{x}$, we want to minimize $C(x) = k\left(3x + \frac{80}{x}\right)$ for $x > 0$. Now $C'(x) = 0 \Rightarrow 3 = \frac{80}{x^2}$ or $x = \sqrt{\frac{80}{3}}$. Since $\lim_{x\to 0^+} C(x) = \infty = \lim_{x\to\infty} C(x)$, this must be a minimum. The dimensions should be $\sqrt{\frac{80}{3}}$ meters for the expensive side and $\sqrt{60}$ meters for the other side.

24. Using standard symbols, we have $A = \pi r^2$ and $C = 2\pi r$. It follows that $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ and $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$, which together show that $\frac{dA}{dt} = r\frac{dC}{dt}$. When $A = 25$, we know $r = \frac{5}{\sqrt{\pi}}$. Therefore

$$\left.\frac{dC}{dt}\right|_{A=25} = \frac{\sqrt{\pi}}{5}\cdot 4 = \frac{4}{5}\sqrt{\pi} \text{ cm/sec.}$$