1. The kernel of T is $\{(x,y,z): T(x,y,z) = (0,0)\}$. We thus need to solve

$$\begin{aligned} 2x - 3y &= 0 \\ x + z &= 0 \end{aligned} \iff \begin{aligned} y &= \tfrac{2}{3}x \\ z &= -x \end{aligned}$$

Letting $x = 3t$, we find that $\ker T = \{(3t, 2t, -3t): t \in \mathbb{R}\}$, revealing that $\ker T$ is a one-dimensional subspace of $\mathbb{R}^3$.

2. We first find the eigenvalues of the matrix.

$$\det\begin{bmatrix} -2-\lambda & -2 \\ -5 & 1-\lambda \end{bmatrix} = (\lambda+2)(\lambda-1) - 10 = \lambda^2 + \lambda - 12 = (\lambda+4)(\lambda-3)$$

The eigenvalues are -4 and 3.

$$\lambda = -4 \qquad \begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \text{eigenvector } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3 \qquad \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} \qquad \text{eigenvector } \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Two linearly independent eigenvectors for the matrix are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ corresponding to the eigenvalues -4 and 3, respectively.

3. By definition, $\ker T = \{u \in U : T(u) = 0\}$. For any scalar c and any two vectors u_1 and u_2 in $\ker T$, we find that

$$T(cu_1) = cT(u_1) = c \cdot 0 = 0,$$
$$T(u_1 + u_2) = T(u_1) + T(u_2) = 0 + 0 = 0,$$

using the properties of linear transformations. It follows that $\ker T$ is a subspace of U.

4. Several options are

i) $\det A \neq 0$

ii) the columns of A are linearly independent

iii) the null space of A contains the zero vector only

iv) the range of the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) = Ax$ is all of $\mathbb{R}^n$

5. We will use row operations.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right]$$

check $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ -3 & 1 & 1 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The inverse of the matrix is $\begin{bmatrix} -2 & 0 & 1 \\ -3 & 1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$.

6. Using Gaussian elimination, we find that

$$\begin{bmatrix} 1 & 2 & 9 & 3 \\ 2 & -1 & 4 & 1 \\ -4 & 7 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 9 & 3 \\ 0 & -5 & -14 & -5 \\ 0 & 15 & 42 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 17/5 & 1 \\ 0 & 1 & 14/5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = 1 - \frac{17}{5}z$

$y = 1 - \frac{14}{5}z$

$z = 5t,\ t \in \mathbb{R}$

The solution set for the system of equations consists of

$x = 1 - 17t$

$y = 1 - 14t$, where t is any real number.

$z = 5t$

7. Suppose that $\{u_1, u_2, \ldots, u_n\}$ spans U. Let $v \in V$. Since T is onto, there exists a vector $u \in U$ such that $T(u) = v$. Since $\{u_1, u_2, \ldots, u_n\}$ spans U, there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n.$$

Since T is linear,

$$v = T(u) = c_1 T(u_1) + c_2 T(u_2) + \cdots + c_n T(u_n),$$

which shows that v is in the span of $\{T(u_1), T(u_2), \ldots, T(u_n)\}$. Since v was an arbitrary element of V, the set $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ spans V.

8. Suppose that y_1 and y_2 are in the set W. This means that $2y_1'' + xy_1 = 0$ and $2y_2'' + xy_2 = 0$. We must show that $y_1 + y_2$ is in W and that cy_1 is in W for any real number c. Since

$$2(y_1 + y_2)'' + x(y_1 + y_2) = 2y_1'' + xy_1 + 2y_2'' + xy_2 = 0 + 0 = 0,$$

$$2(cy_1)'' + x(cy_1) = c(2y_1'' + xy_1) = c \cdot 0 = 0,$$

we see that $y_1 + y_2$ and cy_1 are in W. It follows that W is a subspace of $\mathcal{C}^2[0,1]$.

9. Suppose that $a(x-y)+b(y-z)+c(x+z)=0$. Then $(a+c)x+(b-a)y+(c-b)z=0$. Since $\{x, y, z\}$ is a linearly independent set, the scalars $a+c$, $b-a$, and $c-b$ must all be 0. Note that

$$\begin{matrix} a+c=0 \\ b-a=0 \\ c-b=0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} a+c=0 \\ a=b \\ c=b \end{matrix} \quad \Rightarrow \quad 2b=0$$

and it follows that a, b, and c are all zero. By definition, the set $\{x-y, y-z, x+z\}$ is linearly independent.

10. By inspection, a basis for the column space of A is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ and a basis for the row space of $A = \left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right\}$.

To determine the null space of A (which must have dimension $4-1=3$) we solve $Ax=0$. This reduces to solving $x_1+2x_2+3x_3+4x_4=0$, which has 3 free variables. A basis for the null space of A is thus

$$\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\0\\1\end{bmatrix}\right\}.$$

11. Suppose first that the columns of A are linearly independent. Note that $A^T A$ is an $n \times n$ matrix. To prove that $A^T A$ is nonsingular, we will prove that the only solution to $A^T A x = 0$ is the trivial solution. Suppose that $A^T A x = 0$ for some $x \in \mathbb{R}^n$. Then

$$0 = x^T (A^T A x) = (x^T A^T)(Ax) = (Ax)^T Ax = \|Ax\|^2.$$

Thus $Ax = 0$ and, since the columns of A are linearly independent, we must have $x = 0$. This shows that $A^T A$ is nonsingular.

Now suppose that $A^T A x$ is nonsingular. If $Ax = 0$, then $A^T A x = A^T 0 = 0$ and thus $x = 0$ since the only solution to $A^T A x = 0$ is the trivial solution. We have thus shown that the trivial solution is the only solution to $Ax = 0$ so the columns of A are linearly independent.

12. If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$. If $\lambda = 0$, then $\det A = 0$ and A does not have an inverse. So we know $\lambda \neq 0$. Let v be a nonzero eigenvector corresponding to λ. Then

$$Av = \lambda v \Rightarrow A^{-1}(Av) = A^{-1}(\lambda v) \Rightarrow v = \lambda A^{-1} v \Rightarrow A^{-1} v = \frac{1}{\lambda} v.$$

It follows that $1/\lambda$ is an eigenvalue of A^{-1}.

13. We must prove that $u \cdot v = u^T v = 0$. Let λ and μ be the eigenvalues for u and v, respectively. Thus $Au = \lambda u$, $Av = \mu v$, and $A^T = A$ since A is symmetric. Using these facts, we find that

$$\lambda u^T v = (\lambda u)^T v = (Au)^T v = (u^T A^T) v = u^T (Av) = u^T(\mu v) = \mu u^T v$$

and thus $(\lambda - \mu) u^T v = 0$. Since λ and μ are distinct, it follows that $u^T v = 0$. We conclude that u and v are orthogonal.

14. Let A and B be skew-symmetric matrices and let c be a real number. Then

$$(A+B)^T = A^T + B^T = -A - B = -(A+B) \text{ and}$$

$$(cA)^T = cA^T = -cA,$$

which shows that $A+B$ and cA are skew-symmetric. This shows that the set of all $n \times n$ skew-symmetric matrices is a subspace of $M_{n \times n}$.

A generic 3×3 skew-symmetric matrix has the form $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ so a basis for the subspace of all 3×3 skew-symmetric matrices is

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.$$

15. Since u and v are orthogonal, we know that $u \cdot v = 0$. Suppose that there are scalars c and d such that $cu + dv = 0$. We then have

$$0 = u \cdot (cu + dv) = c\, u \cdot u + d\, u \cdot v = c\|u\|^2 \text{ and}$$
$$0 = v \cdot (cu + dv) = c\, v \cdot u + d\, v \cdot v = d\|v\|^2.$$

Since u and v are not the zero vector, the numbers $\|u\|^2$ and $\|v\|^2$ are nonzero. This means that $c = 0$ and $d = 0$. By definition, the vectors u and v are linearly independent.

16. a) A set of m vectors in $\mathbb{R}^n$ with $m > n$ is linearly dependent.

b) If A is an $m \times n$ matrix, then the matrix $A^T A$ is an $n \times n$ matrix.

c) Similar matrices have the same eigenvalues.

d) An orthogonal set of nonzero vectors is linearly independent.

17. Since A and B are similar, there exists a matrix P such that $B = P^{-1}AP$. We then have

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}AP - P^{-1}(\lambda I)P) \\ &= \det(P^{-1}(A - \lambda I)P) \\ &= \det P^{-1} \cdot \det(A - \lambda I) \cdot \det P \\ &= \det(A - \lambda I). \end{aligned}$$

Since A and B have the same characteristic polynomial, they have the same eigenvalues.

18. For a given $m \times n$ matrix A, we know that

$$\dim(\text{row } A) = \dim(\text{col } A) \text{ and } \dim(\text{col } A) + \dim(\text{nul } A) = n.$$

Since our matrix A has linearly independent rows, we know that $\dim(\text{row } A) = 24$. It follows that $\dim(\text{nul } A) = 60 - 24 = 36$.

19. The null space of A is the set of all vectors x in $\mathbb{R}^p$ for which $Ax = 0$. The column space of B is the set of all vectors x in $\mathbb{R}^p$ for which $By = x$ for some y in $\mathbb{R}^n$. Suppose that x is in the column space of B and choose y in $\mathbb{R}^n$ so that $x = By$. Since AB is the zero matrix $Ax = A(By) = (AB)y = 0$, showing that x is in the null space of A. Since x was an arbitrary vector in col B, we find that col $B \subseteq$ nul A.

20. Given two elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ of $M_{2\times 2}$ and a real number r, we find that

$$T\left(r\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}\right) = (ra + rd)t + rb$$
$$= r\big((a+d)t + b\big) = r\,T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right);$$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = T\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) = (a+e+d+h)t + b + f$$
$$= (a+d)t + b + (e+h)t + f = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right).$$

Hence, the function T is a linear transformation. Matrices of the form $\begin{bmatrix} a & 0 \\ c & -a \end{bmatrix}$ are sent to the zero polynomial, so a basis for the kernel of T is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

21. Since P_2 has dimension 3, a linearly independent set of 3 vectors in P_2 forms a basis for P_2. To see if the given set is linearly independent, suppose that scalars a, b, and c satisfy

$$a(t^2 - t + 3) + b(-2t^2 + 4t) + c(t^2 + 5t - 6) = 0$$

It follows that

$$(a - 2b + c)t^2 + (-a + 4b + 5c)t + (3a - 6c) = 0$$

and thus

$$\begin{array}{l} a - 2b + c = 0 \\ -a + 4b + 5c = 0 \\ 3a \quad\;\; - 6c = 0 \end{array} \Rightarrow \begin{array}{l} -2b + a + c = 0 \\ 4b + 5c - a = 0 \\ a = 2c \end{array} \Rightarrow \begin{array}{l} -2b + 3c = 0 \\ 4b + 3c = 0 \end{array} \Rightarrow 6b = 0$$

Hence, all three scalars must be 0. Since the set $\{t^2 - t + 3, -2t^2 + 4t, t^2 + 5t - 6\}$ is linearly independent, it forms a basis for P_2.

22. Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent set in V. We must prove that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a linearly independent set in W. Suppose that

$c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = 0$ Then

$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = 0$ since T is linear and

$c_1 v + c_2 v_2 + \cdots + c_n v_n = 0$ since T is one-to-one.

Since $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, all the scalars must be 0. Therefore the set $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

23. The rank of a matrix is the dimension of its column space. Since B and AB are both $n \times p$ matrices, we know that

$$\dim(\operatorname{col} B) + \dim(\operatorname{nul} B) = p = \dim(\operatorname{col} AB) + \dim(\operatorname{nul} AB).$$

The proof is thus complete if we show that AB and B have the same null spaces. Note that

$$Bx = 0 \Rightarrow A(Bx) = 0 \Rightarrow (AB)x = 0 \quad \Rightarrow \operatorname{nul} B \subseteq \operatorname{nul} AB;$$

$$(AB)x = 0 \Rightarrow A^{-1}(AB)x = 0 \Rightarrow Bx = 0 \Rightarrow \operatorname{nul} AB \subseteq \operatorname{nul} B;$$

where we have used the fact that A is invertible for the second implication. Thus shown that $\operatorname{nul} B = \operatorname{nul} AB$ and thus $\dim(\operatorname{col} B) = \dim(\operatorname{col} AB)$, that is, the matrices B and AB have the same rank.

24. Row reducing the corresponding matrix yields

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -1 & -1 & 1-a \\ 0 & 1 & 1 & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 1 & a-1 \\ 0 & 0 & 0 & 1-a+b \end{array}\right].$$

In order for the system of equations to be consistent (and thus have solutions) is for

$1 - a + b = 0$ or $a - b = 1$.