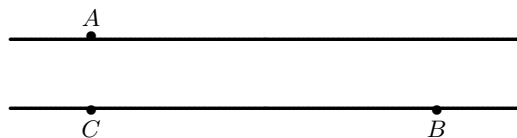


## Calculus I questions

- (a) State the Mean Value Theorem.  
  
(b) Discuss both the hypotheses and the conclusion of this theorem for the function  $f(x) = x^{2/3}$  on the interval  $[-1, 27]$ .  
  
(c) Find the point  $c$  guaranteed by the theorem for the function  $g(x) = (x^2 + 3x)/(x - 1)$  on the interval  $[-1, 0]$ .
- Determine the  $(x, y)$  coordinates of all inflection points that lie on the graph of the function  $f(x) = 4e^{-x^2/10}$ .
- Find an equation for the line that is tangent to the curve  $y = x\sqrt{3x + 1}$  at  $x = 1$ .
- Find an equation of a line that is tangent to the curve  $y = x^2$  and passes through the point  $(0, -8)$ .
- Give an example of a function  $f$  such that  $\lim_{x \rightarrow \infty} (3x^2 + 4x - 600)/f(x) = 3/4$ . Show that your choice of  $f$  has the above property. Is  $f$  unique? Explain.
- A rectangular box with a square base is to be constructed using materials that cost one dollar per square foot for the top, two dollars per square foot for the sides, and four dollars per square foot for the base. Find the volume of the largest box that can be constructed for eighty dollars.
- Suppose that a particle is moving in a straight line with a velocity of 120 ft/sec. At time  $t = 0$ , the object begins to decelerate at the rate of  $-6t$  ft/sec<sup>2</sup>. How far does this particle travel before coming to a stop?
- Let  $f$  be the function defined by  $f(x) = 1/x$  and let  $[a, b]$  be an interval with  $a > 0$ . Find the point  $c$ , in simplified form, guaranteed by the Mean Value Theorem for  $f$  on  $[a, b]$ .
- Any line with negative slope that passes through the point  $(5, 2)$  forms a triangle with the positive  $x$  and  $y$  axes. Among all such lines, find the line that gives the triangle with the least area. Include an argument that shows that you have actually found the minimum area.
- An underground cable must connect point  $A$  with point  $B$  (see the figure). The distance from  $A$  to  $C$ , which is across a marshy bog, is 80 feet while the distance from  $C$  to  $B$ , which is level solid ground, is 400 feet. It costs \$100 per foot to lay the cable in the bog and \$50 per foot to lay the cable in solid ground. What path for the cable will minimize the total cost?



- A balloon leaves the ground 200 feet from a person standing at ground level. When the balloon is at a height of 80 feet, it is rising at the rate of 30 ft/min. How fast is the person's angle of observation increasing at this instant?
- For the curve defined by  $x^2 + 3xy - 2y^2 - x + 5y = 11$ , find  $\frac{dy}{dx}\bigg|_{(2,1)}$ .

## Calculus II questions

1. A cylindrical hole of radius  $a$  is bored through the center of a sphere of radius  $r > a$ . Find the volume of the solid that remains.
2. Find the total area of the region bounded by the curve  $y = x(x - 2)(x - 5)$  and the  $x$ -axis.
3. Evaluate  $\int \frac{x^3}{x^2 + 4} dx$ .
4. State both versions of the Fundamental Theorem of Calculus. Write down a function  $f$  such that  $f'(x) = 2^x/x$  and  $f(1) = 0$ .
5. Let  $A$  be the region that is under the graph of  $y = \cos x$  and above the  $x$ -axis on the interval  $[0, \pi/2]$ . Find the volume of the solid that is generated when  $A$  is revolved around the  $y$ -axis.
6. Find the sum of the series  $\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^{n-1}}$ .
7. Find the radius of convergence **and** the interval of convergence for the power series  $\sum_{n=1}^{\infty} \frac{5}{n3^n} (x - 2)^n$ . Then let  $f$  be the function defined by this power series on its interval of convergence and find  $f^{(10)}(2)$ .
8. Evaluate any two of the following integrals. Choose wisely.  
(i)  $\int x \sin(x^2) dx$                       (ii)  $\int x^2 \sin(x^2) dx$                       (iii)  $\int x \sin(x) dx$
9. Determine whether or not the series  $\sum_{k=1}^{\infty} \left(\frac{k}{3k+1}\right)^k$  converges.
10. Let  $R$  be the region under the graph of  $f(x) = 1/x$  and above the  $x$ -axis on the interval  $[1, \infty)$ .
  - a. Is the area of  $R$  finite? If so, what is its value?
  - b. Suppose that  $R$  is revolved around the  $x$ -axis. Is the volume of this solid finite? If so, what is its value?
11. Evaluate  $\int_1^{\sqrt{2}} \frac{\sqrt{x^2 - 1}}{x^4} dx$ .
12. Find the limit of the sequence  $\{k(\sqrt[k]{10} - 1)\}$ .
13. Suppose that the following table represents the velocity of a particle moving in a straight line.

$t$	(sec)	0	1	2	3	4	5	6
$v$	(m/sec)	0	5	12	15	14	6	0

Use Simpson's rule to approximate the distance traveled by the particle.

### Calculus III questions

1. Find the directional derivative of  $f(x, y, z) = x^2y - xyz$  at the point  $(1, 2, 1)$  in the direction of the point  $(3, 3, -2)$ .
2. A function  $u$  of two variables is said to be harmonic if  $u_{xx} + u_{yy} = 0$ . Determine whether or not the function  $u(x, y) = \ln(x^2 + y^2)$  is harmonic.
3. Evaluate the double integral  $\int_0^\pi \int_{y/2}^{\pi/2} \frac{\sin x}{x} dx dy$ .
4. Find an equation for the plane that contains the point  $(1, 0, 1)$  and the line with parametric equations  $x = 1 - t, y = 2 + t, z = t$ .
5. Evaluate  $\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \sqrt{x^2 + y^2} dy dx$ .
6. Find the maximum value of the function  $f(x, y, z) = 2x - 2y + z$  subject to the condition that  $x^2 + y^2 + z^2 = 9$ .
7. Let  $\ell_1$  be the tangent line to the curve  $y = x^2$  at the point  $(1, 1)$  and let  $\ell_2$  be the tangent line to the curve  $y = x^3$  at the point  $(1, 1)$ . Find the cosine of the acute angle formed by  $\ell_1$  and  $\ell_2$ .
8. Let  $\mathbf{F}$  be the vector field defined by  $\mathbf{F}(x, y, z) = (3y^3 - 10xz^2)\mathbf{i} + 9xy^2\mathbf{j} - 10x^2z\mathbf{k}$  and let  $C$  be the path given by  $\mathbf{r}(t) = \langle 1 + \sin(\pi t), 2 + \cos(2\pi t), 3 + \cos(3\pi t) \rangle$  for  $0 \leq t \leq 1$ . Find the work done by the field  $\mathbf{F}$  in moving a particle along the path  $C$ .
9. Find the volume of the solid bounded by the planes  $y = 0, x = 2, y = x, z = 0$ , and  $z = 2x + 2y + 5$ .
10. Let  $\mathbf{F}$  be the vector field defined by  $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$  and let  $C$  be the path that consists of the straight line from  $(0, 0, 0)$  to  $(1, 1, 0)$  followed by the vertical line from  $(1, 1, 0)$  to  $(1, 1, 1)$ . Find the work done by the field  $\mathbf{F}$  in moving a particle along the path  $C$ .
11. Find a point on the surface  $z = x^2y - 4xy + 5x - 8$  where the tangent plane is perpendicular to the line  $x = 1 + 6t, y = -3 - 10t, z = 2t$ .
12. A flat circular plate has the shape of the region  $\{(x, y) : x^2 + y^2 \leq 1\}$ . The plate is heated so that the temperature at any point  $(x, y)$  is  $T(x, y) = x^2 + 2y^2 - x + 100$ . Find the hottest and coldest points on the plate and the temperature at these points.
13. Find parametric equations for the line of intersection of the planes  $2x + y + z = 4$  and  $3x - y + z = 3$ .
14. Find the work done by the force field  $\mathbf{F}(x, y) = \langle -y^2, 2x \rangle$  in moving a particle from  $(2, 0)$  to  $(-2, 0)$  along the upper half of the circle  $x^2 + y^2 = 4$ .

## Linear Algebra questions

1. Let  $A$  be an  $n \times n$  matrix and suppose that 2 is an eigenvalue of  $A$ . Prove that 2 is an eigenvalue of the matrix  $A^2 - A$ .
2. Consider the linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  $T(x, y, z) = (2x - 3y, x + z)$ . Find the kernel of  $T$ .
3. Find a basis for the set of all vectors  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  in  $\mathbb{R}^4$  such that  $x_1 = x_3$  and  $x_2 = -x_4$ . Justify your response and be certain to express your answer using correct notation.
4. Find two linearly independent eigenvectors for the matrix  $\begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$
5. Let  $U$  and  $V$  be vector spaces and let  $T : U \rightarrow V$  be a linear transformation. Prove that the kernel of  $T$  is a subspace of  $U$ .
6. This problem considers the inverse of a matrix.
  - a) Let  $A$  be an  $n \times n$  matrix. Write down three different statements that are equivalent to the statement “ $A$  is invertible”. (You need three statements besides “ $A$  is invertible” and they should all be qualitatively different.)
  - b) Find the inverse of the matrix  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$ .
7. Find all solutions to the system
$$\begin{aligned}x + 2y + 9z &= 3 \\ 2x - y + 4z &= 1 \\ -4x + 7y + 6z &= 3\end{aligned}$$
8. Write down a basis for the set of all  $3 \times 3$  symmetric matrices.
9. Let  $T : U \rightarrow V$  be a linear transformation of  $U$  onto  $V$ . Prove that  $\{T(u_1), \dots, T(u_n)\}$  spans  $V$  if  $\{u_1, \dots, u_n\}$  spans  $U$ .
10. Recall that  $\mathcal{C}^2([0, 1])$  is the vector space of all real-valued functions whose second derivative is continuous on  $[0, 1]$ . Prove that  $W = \{y \in \mathcal{C}^2([0, 1]) : 2y'' + xy = 0\}$  is a subspace of  $\mathcal{C}^2([0, 1])$ .
11. Suppose that  $\{x, y, z\}$  is a linearly independent set of vectors in some vector space  $V$ . Determine whether or not the set  $\{x - y, y - z, x + z\}$  is linearly independent.
12. Find a basis for the null space of  $A$  and a basis for the column space of  $A$  if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$