Chapter 4

Hyperbolic Plane Geometry

"It is quite simple, ... , but the way things come out of one another is quite lovely." — Clifford

36. Introduction.

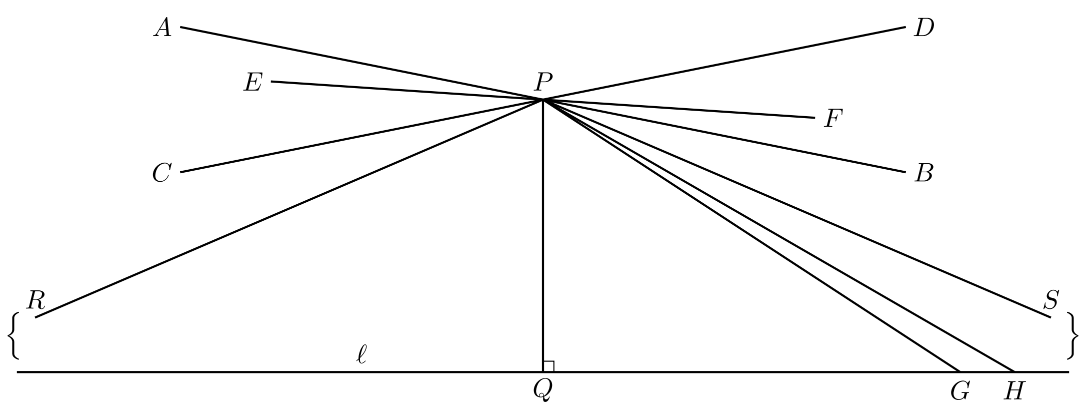
There is much to be said in favor of carefully formulating, at this point, a set of explicitly stated assumptions as a foundation for the study of the geometry of the Hyperbolic Plane.[1] For very mature students and for those who have been over the ground before, this is doubtless the best procedure. But for others, such a precise, rigorously logical treatment would only prove confusing. For this reason, keeping in mind our objectives, we propose to follow the path of the pioneers, to avail ourselves of the familiar foundations of Euclidean Geometry, replacing the Fifth Postulate by a contradictory one, and making such other changes as may consequently be forced upon us. Thus all of the Euclidean propositions which do not depend upon the Fifth Postulate, in particular the first twenty-eight, are immediately available. Nor are simplicity and economy the only advantages of such an approach. This presentation of the material, in the way it was acquired, has, we believe, the sanction of sound pedagogical and psychological principles. Refinement and extreme rigor may very well come later.

37. The Characteristic Postulate of Hyperbolic Geometry and Its Immediate Consequences.

In Euclidean Plane Geometry, the Fifth Postulate is essentially equivalent to the statement that through a given point, not on a given line, one and only one line can be drawn which does not intersect the given line. In its place we introduce the following as the *Characteristic Postulate* of Hyperbolic Plane Geometry.

Postulate. *Through a given point, not on a given line, more than one line can be drawn not intersecting the given line.*

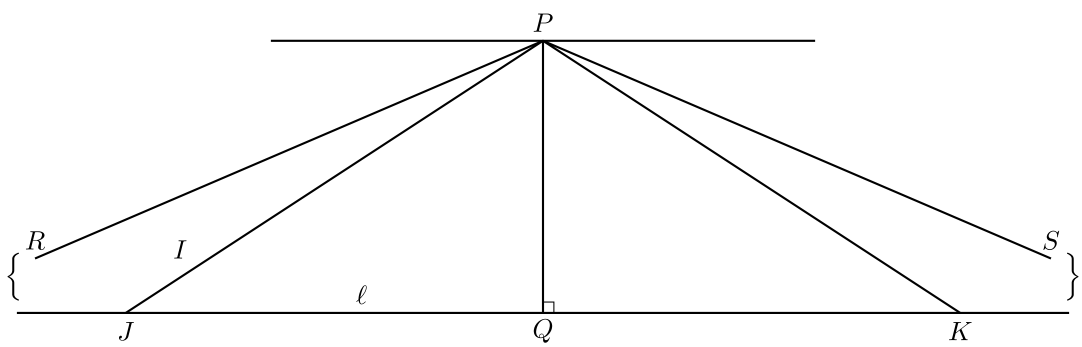
We have already observed that, if there is more than one line through the given point not intersecting the given line, there are an infinite number of them. If P is the given point, ℓ the given line, and AB and CD two lines through P which do not intersect ℓ, then no line, such as EF, lying within the vertical angles APC and DPB, which do not contain the perpendicular PQ from P to ℓ, will cut ℓ.



For if EF, produced for example to the right, were to cut ℓ, then, since PF and the perpendicular PQ would both intersect ℓ, PB would also have to intersect ℓ by the Axiom of Pasch.

If one starts with the perpendicular PQ from P to ℓ and allows PQ to rotate about P in either direction, say counterclockwise, it will continue to intersect ℓ awhile and then cease to intersect it. Thus a situation is reached in which the lines through P are divided into two sets, those which cut ℓ and those which do not, each line of the first set preceding each line of the second. Under these circumstances, the Postulate of Dedekind asserts that there exists a line through P which brings about this division of the lines into the two sets. Since this line itself either cuts ℓ or does not cut it, it must either be the last of the lines which intersect ℓ or the first of those which do not. But there is no last cutting line. For, if one assumes that PG is the last of the cutting lines, and measures off any distance GH on the side of G opposite Q, then PH is a cutting line and a contradiction has been reached. Hence the dividing line is the first of those which do not intersect ℓ. A similar situation is encountered if PQ is rotated clockwise. Thus there are two lines PR and PS, through P, which do not cut ℓ, and which are such that every line through P, lying within the angle RPS, does cut ℓ.

Furthermore, the angles RPQ and SPQ are equal. If they are not, one of them is the greater, say RPQ. Measure angle IPQ equal to angle SPQ [by I.23].



Then PI will cut ℓ in a point J. Measure off on ℓ, on the side of Q opposite J, QK equal to QJ. Draw PK. It follows from congruent right triangles [SAS] that angle QPK equals angle QPJ and hence angle QPS. But PS does not intersect ℓ and a contradiction has been reached. One concludes that angles RPQ and SPQ are equal.

It can easily be shown that these two angles are acute. If they were right angles, PR and PS would lie on the same straight line and this line would be the line through P perpendicular to PQ. But that perpendicular does not intersect ℓ (Euclid I.28), and furthermore it is not the only line through P which does not intersect ℓ. Consequently there will be lines through P within the angle RPS which do not cut ℓ under these circumstances and again a contradiction is encountered.

These results can be summarized in the following theorem:

Theorem 1: If ℓ is any line and P is any point not on ℓ, then there are always two lines through P which do not intersect ℓ, which make equal acute angles with the perpendicular from P to ℓ, and which are such that every line through P lying within the angle containing that perpendicular intersects ℓ, while every other line through P does not.

All of the lines through P which do not meet ℓ are parallel to ℓ from the viewpoint of Euclid. Here, however, we wish to recognize the peculiar character of the two described in the theorem above. These two lines we call the two *parallels* to ℓ through P, and designate the others as *non-intersecting* with reference to ℓ. We shall discover shortly that the size of the angle which each of the two parallels makes with the perpendicular from P to ℓ depends upon the length h of this perpendicular. The angle is called the *angle of parallelism* for the distance h and will be denoted by $\Pi(h)$ in order to emphasize the functional relationship between the angle and the distance. On occasion it will be found possible and convenient to distinguish between the two parallels by describing one as the *right-hand*, the other as the *left-hand*, parallel.

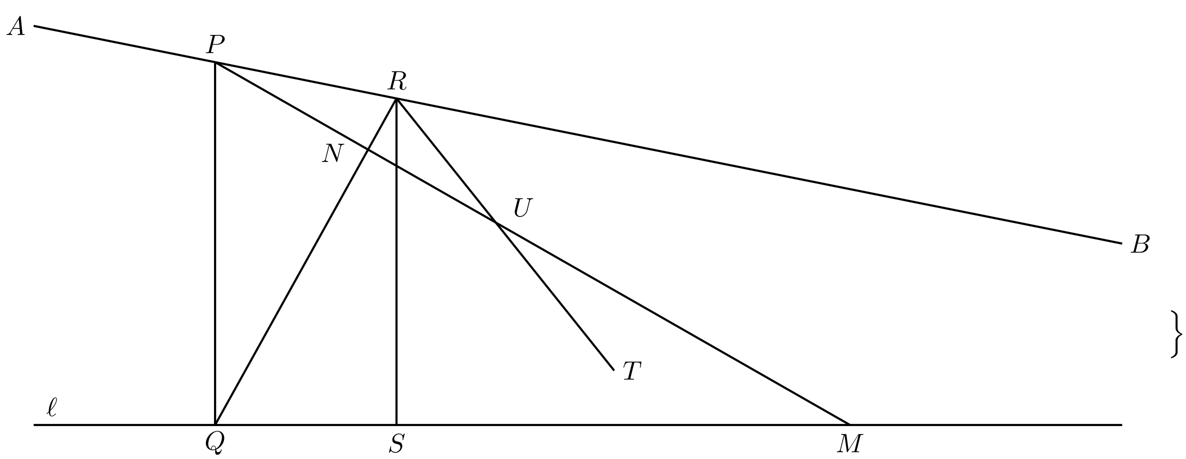
Exercises

1. If two lines BA and BC are both parallel to line ℓ, show that the bisector of angle ABC is perpendicular to ℓ.

38. Elementary Properties of Parallels. Certain properties of Euclidean parallels hold also for parallels in Hyperbolic Geometry. Three of these are described in the following theorems.

Theorem 2: If a straight line is the parallel through a given point in a given sense to a given line, it is, at each of its points, the parallel in the given sense to the given line.

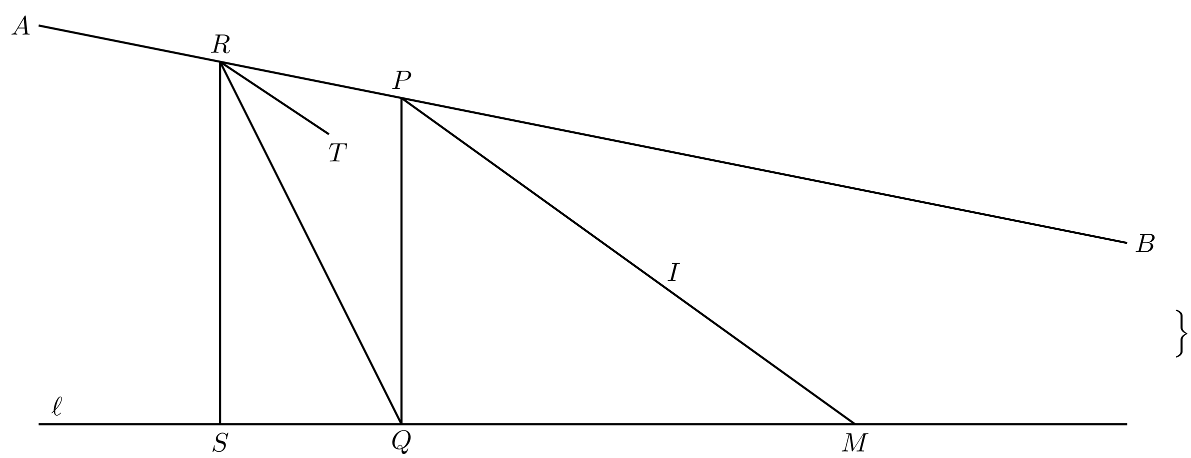
Statement	Reason
1. $AB \parallel \ell$ toward the right from P	1. given
Case I	
2. R is on AB to the right of P	2. added premise



Statement	Reason
3. $PQ \perp \ell$, $RS \perp \ell$	3. constructed (I.12)
4. line RT lies within $\angle SRB$	4. added premise
5. U is a point on line RT	5. lines contain points
6. PU intersects line ℓ in point M	6. PU lies within $\angle QPB$
7. PU intersects QR in point N	7. Axiom of Pasch for $\triangle QPR$
8. RU intersects QM	8. Axiom of Pasch for $\triangle QNM$
9. RT intersects ℓ	9. follows from (8)

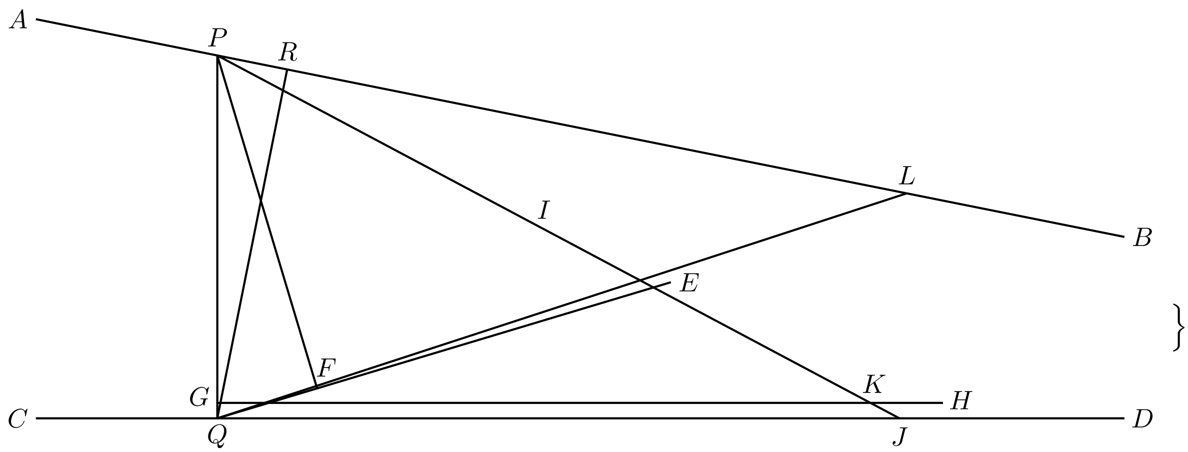
Case II

10. R is on AB to the left of P — 10. added premise



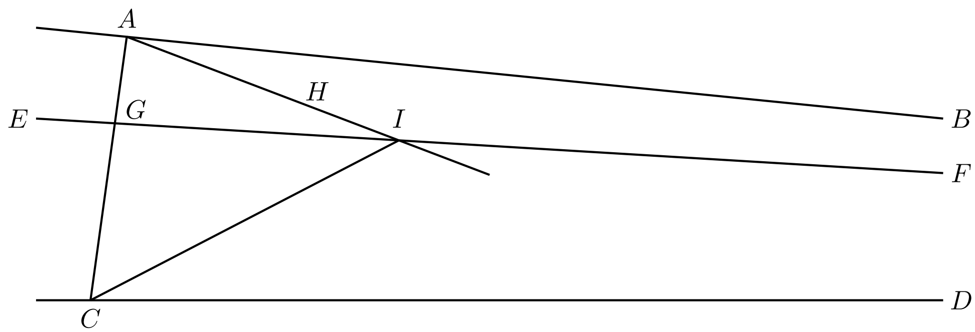
11. $PQ \perp \ell, \quad RS \perp \ell$ — 11. constructed (I.12)
12. RT lies within $\angle QRP$ (within $\angle SRQ$ is trivial) — 12. added premise
13. RT intersects PQ — 13. Axiom of Pasch for $\triangle QRP$
14. $\angle TRP < \angle QPB$ — 14. I.16
15. $\angle BPI = \angle TRP$ — 15. constructed (I.23)
16. PI intersects ℓ at M — 16. Theorem 1
17. RT does not intersects PM — 17. I.28
18. RT intersects QM — 18. Axiom of Pasch for $\triangle QPM$
19. RT intersects ℓ — 19. follows from (18)

Theorem 3: If one line is parallel to a second line, then the second line is parallel to the first line.

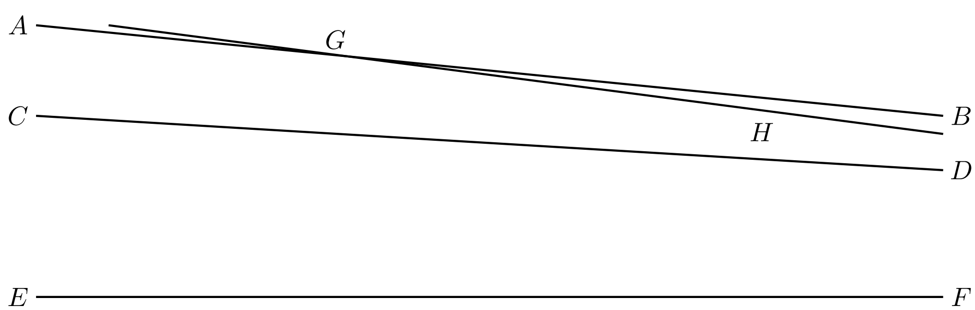


Statement	Reason
1. $AB \parallel CD$ toward the right from P	1. given
2. $PQ \perp CD$, $QR \perp AB$, R is to the right of P	2. constructed (I.12), I.16
3. E is an arbitrary point inside $\angle RQD$	3. assumed
4. $PF \perp QE$	4. constructed (I.12)
5. F and E are on the same side of PQ	5. I-16
6. $PG = PF$, note $PG < PQ$	6. constructed (I.3), I.19 for $\triangle PQF$
7. $GH \perp PQ$	7. constructed (I.11)
8. $\angle GPI = \angle FPB$	8. constructed (I.23)
9. PI intersects CD at J	9. Theorem 1 ($\angle QPI < \angle QPB$)
10. GH cuts PQ, GH does not cut CD	10. by construction, I.28
11. GH cuts PJ at K	11. Axiom of Pasch for $\triangle PQJ$
12. $PL = PK$, join F and L	12. constructed (I.3)
13. $\triangle GPK \cong \triangle FPL$	13. SAS
14. $\angle PFL$ is a right angle	14. CPCTE
15. QE and QL are the same line	15. both are perpendicular to PF
16. QE intersects AB at L	16. by statement 15
17. $QD \parallel AB$ toward the right from Q	17. point E was arbitrary
18. $CD \parallel AB$ toward the right from C	18. Theorem 2 (note that Q was not arbitrary)

Theorem 4: If two lines are both parallel to a third line in the same direction, then they are parallel to one another.



Statement	Reason
1. $AB \parallel EF$, $CD \parallel EF$ toward the right	1. given, WLOG AB is "above" CD
Case I (EF is between AB and CD)	
2. AC cuts EF at G	2. A and C on opposite sides of EF
3. AH is a line within $\angle CAB$	3. added premise
4. AH cuts EF in a point I	4. AH falls within $\angle GAB$
5. $EF \parallel CD$ toward the right	5. Theorem 3
6. $IF \parallel CD$ toward the right	6. Theorem 2
7. AH intersects CD	7. $AH = AI$ falls within $\angle CIF$
8. AB does not cut CD	8. AB does not cut EF
9. $AB \parallel CD$ toward the right	9. AH arbitrary line in $\angle CAB$



Statement	Reason
Case II (EF is "below" CD; Case III with EF "above" AB is similar)	
10. assume $AB \nparallel CD$ toward the right	10. added premise
11. take point G on AB	11. lines contain points
12. $GB \nparallel CD$ toward the right	12. Theorem 2
13. through G draw $GH \parallel CD$ toward the right	13. such a parallel exists
14. $GH \neq GB$	14. obvious from (12) and (13)
15. $EF \parallel CD$ toward the right	15. Theorem 3 given $CD \parallel EF$
16. $GH \parallel EF$ toward the right	16. Case I
17. $GB \parallel EF$ toward the right	17. Theorem 2 given $AB \parallel EF$
18. $GH = GB$	18. parallels are unique
19. $AB \parallel CD$ toward the right	19. proof by contradiction ((14) and (18))

39. Ideal Points.

We wish to introduce at this point an important concept in connection with parallel lines. Two intersecting lines have a point in common, but two parallel lines do not, since they do not intersect. However, two parallel lines do have *something* in common. It is convenient to recognize this relationship by saying that two parallel lines have in common, or intersect in, an *ideal point*. (Also called, more frequently than not, a point at infinity or infinitely distant point.) Thus all of the lines parallel in the same sense to any line, and consequently parallel to one another, will be thought of as being concurrent in an ideal point and constituting a sheaf of lines with an ideal vertex. Every line contains thus, in addition to its *ordinary* or *actual* points, two ideal points through which pass all of the lines parallel to it in the two directions.

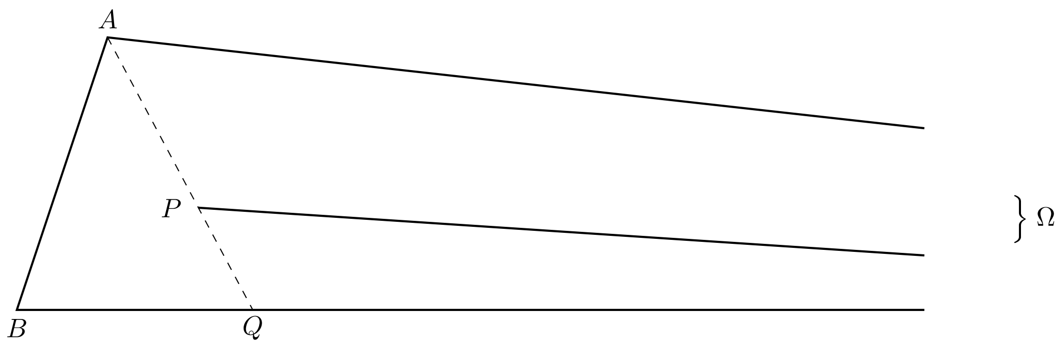
While ideal points are concepts, so also, for that matter, are ordinary points. The introduction of these ideal elements is primarily a matter of convenient terminology. To say that two lines intersect in an ideal point is merely another way of saying that they are parallel; to refer to the line joining an ordinary point to one of the ideal points of a certain line amounts to referring to the line through the ordinary point parallel to that line in the sense designated. But we shall not be surprised to find these new entities assuming more and more significance as we go on. In the history of mathematics can be found more than one example of an idea introduced for convenience developing into a fundamental concept. As a matter of fact, the use of such ideal elements has been an important factor in the development of geometry and in the interpretation of space. We shall return to this later.

It will gradually be recognized that, in so far as we are concerned with purely descriptive properties, we need not discriminate between ordinary and ideal points. Two distinct points, for example, determine a line, regardless of whether both points are ordinary, both ideal, or one ordinary and the other ideal. In no case is this more strikingly illustrated than in that of the triangle with two vertices ordinary points and the third ideal. We study this figure next.

40. Some Properties of an Important Figure.

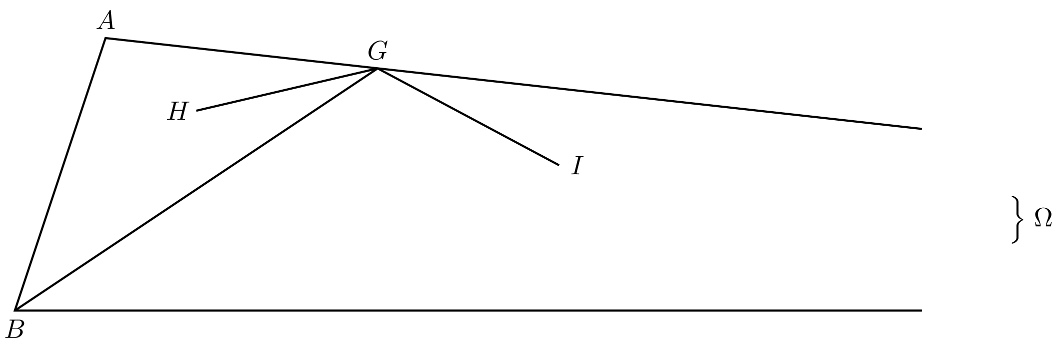
The figure formed by two parallel lines and the segment joining a point of one to a point of the other plays an important role in what is to follow. Let $A\Omega$ and $B\Omega$ be any two parallel lines. Here we follow the convention of using the large Greek letters (generally Ω) to designate ideal points. Let A, any point of the first, and B, any point of the second, be joined. The resulting figure is in the nature of a triangle with one of its vertices an ideal point; it has many properties in common with ordinary triangles. We prove first that Pasch's Axiom holds for such a triangular figure.

Theorem 5: If a line passes within the figure $AB\Omega$ through one of the vertices, it will intersect the opposite side.

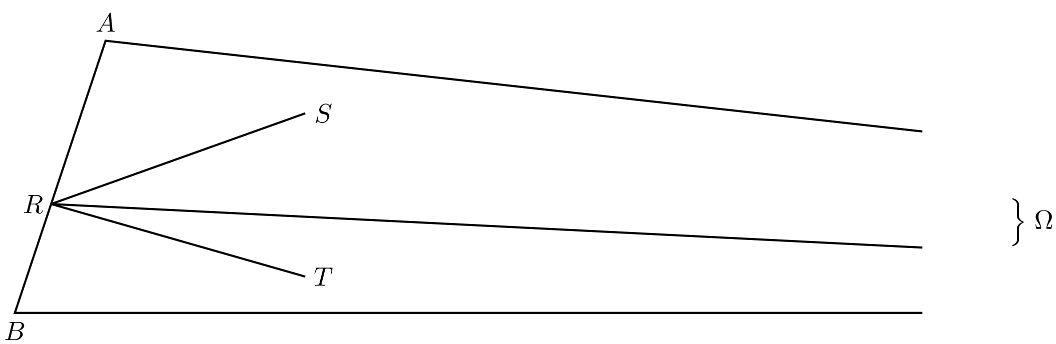


Statement	Reason
1. $A\Omega \parallel B\Omega$ toward the right	1. given
2. point P is inside figure $AB\Omega$	2. given
3. AP intersects $B\Omega$ in Q	3. $\angle BAP$ lies within $\angle BA\Omega$
4. BP intersects $A\Omega$	4. $\angle ABP$ lies within $\angle AB\Omega$
5. $P\Omega \parallel A\Omega$ toward the right	5. parallels exist
6. $P\Omega \parallel B\Omega$ toward the right	6. Theorem 4
7. $P\Omega$ does not intersect BQ	7. follows from 6
8. $P\Omega$ intersects AB	8. Axiom of Pasch for $\triangle ABQ$

Theorem 6: If a straight line intersects one of the sides of $AB\Omega$, but does not pass through a vertex, it will intersect one and only one of the other two sides.

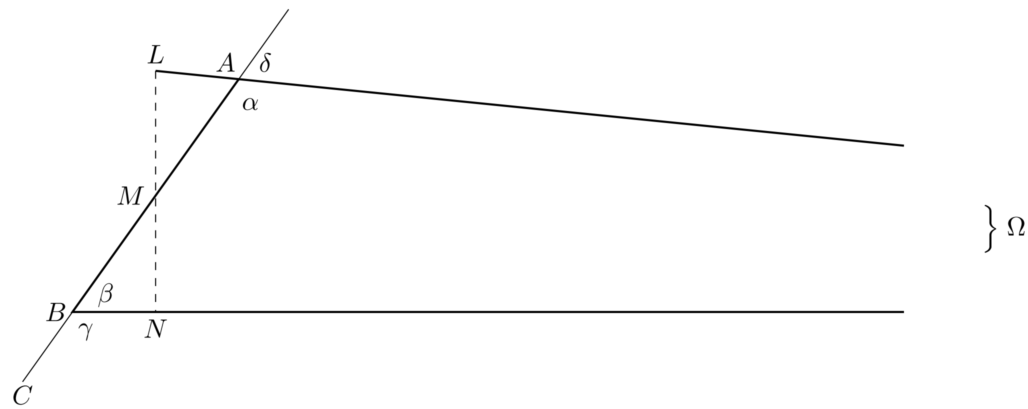


Statement	Reason
1. $A\Omega \parallel B\Omega$ toward the right	1. given
Case I (Case II is similar for $B\Omega$)	
2. line ℓ intersects $A\Omega$ in point G	2. given
3. line $\ell = GH$ intersects AB	3. Axiom of Pasch for $\triangle ABG$
4. $G\Omega \parallel B\Omega$ toward the right	4. Theorem 2
5. line $\ell = GI$ intersects $B\Omega$	5. Theorem 5 applied to $\triangleright BG\Omega$

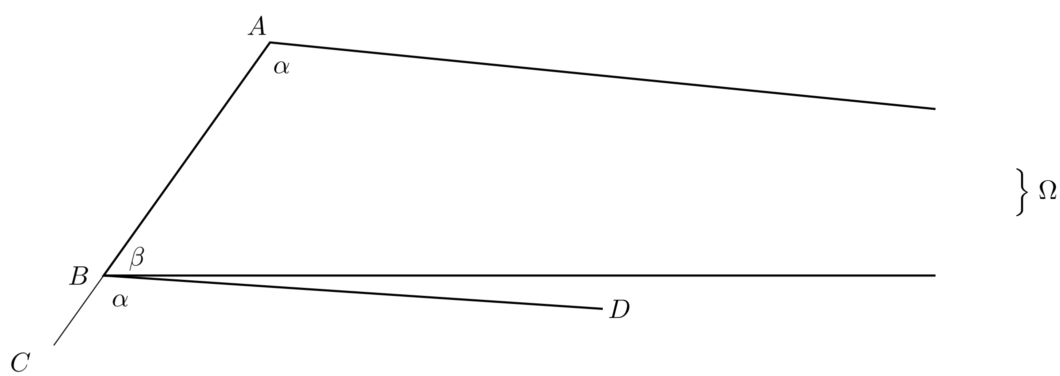


Case III	
6. line ℓ intersects AB in point R	6. given
7. $R\Omega \parallel A\Omega$ toward the right	7. parallels exist
8. $R\Omega \parallel B\Omega$ toward the right	8. Theorem 4
9. line $\ell = RS$ intersects $A\Omega$	9. Theorem 5 applied to $\triangleright RA\Omega$
10. line $\ell = RT$ intersects $B\Omega$	10. Theorem 5 applied to $\triangleright BR\Omega$

Theorem 7: The exterior angles of $AB\Omega$ at A and B, made by producing AB, are greater than their respective opposite interior angles. (In the figure below, $\gamma > \alpha$ and $\delta > \beta$.)



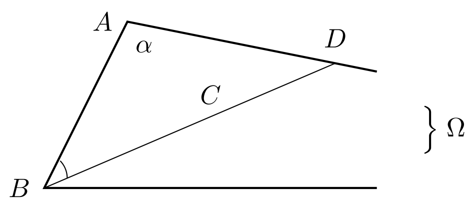
Statement	Reason
1. $A\Omega \parallel B\Omega$ toward the right	1. given
2. assume $\gamma = \alpha$	2. added premise
3. M is the midpoint of AB	3. constructed (I.10)
4. $MN \perp B\Omega$	4. constructed (I.12)
5. $AL = BN$, join M to L	5. constructed (I.3)
6. $\triangle MBN \cong \triangle MAL$	6. SAS
7. $\angle LMA + \angle AMN = \angle BMN + \angle AMN = \pi$	7. CPCTE, I.13
8. LMN is a straight line	8. I.14
9. $L\Omega \parallel B\Omega$ toward the right	9. Theorem 2
10. $\angle NL\Omega = \pi/2$	10. CPCTE
11. $\gamma \neq \alpha$	11. (9) and (10) contradict Theorem 1

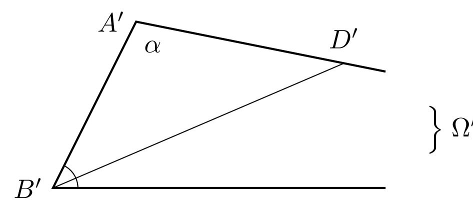


12. $\angle CBD = \alpha$	12. constructed (I.23)
13. BD does not intersect $A\Omega$	13. I.28
14. BD lies within $\angle CB\Omega$	14. Theorem 1
15. $\gamma = \angle CB\Omega > \angle CBD = \alpha$	15. follows from (14)

Next we describe the conditions under which two such figures, $AB\Omega$ and $A'B'\Omega'$, are congruent.

Theorem 8: If AB and $A'B'$ are equal, and angle $BA\Omega$ is equal to angle $B'A'\Omega'$, then angle $AB\Omega$ is equal to angle $A'B'\Omega'$ and the figures are congruent. (SA criterion)

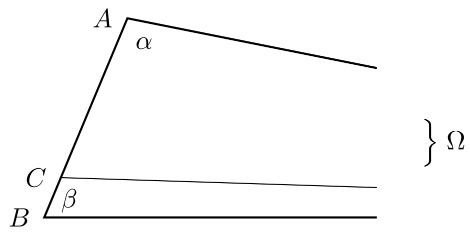


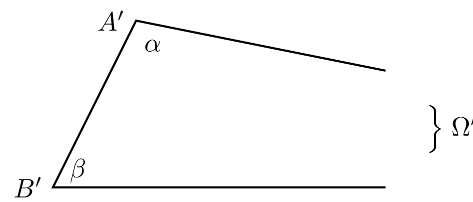


<u>Statement</u>	<u>Reason</u>
1. $AB = A'B'$, $\angle BA\Omega = \angle B'A'\Omega' = \alpha$	1. given
2. $\angle AB\Omega \neq \angle A'B'\Omega'$, WLOG $\angle AB\Omega > \angle A'B'\Omega'$	2. added premise
3. $\angle ABC = \angle A'B'\Omega'$	3. constructed (I.23)
4. BC intersects $A\Omega$ at D	4. Theorem 1
5. select D' on $A'\Omega'$ so that $A'D' = AD$	5. constructed (I.3)
6. $\triangle ABD \cong \triangle A'B'D'$	6. SAS
7. $\angle ABD = \angle A'B'D'$	7. CPCTE
8. $\angle A'B'D' = \angle A'B'\Omega'$	8. since both equal $\angle ABC$
9. $\angle A'B'D' < \angle A'B'\Omega'$	9. $B'D'$ is a cutting line
10. $\angle AB\Omega = \angle A'B'\Omega'$	10. proof by contradiction

It is perhaps superfluous to remark that this theorem still holds if one of the figures is reversed, either by drawing the parallels in the opposite direction or by interchanging the two angles.

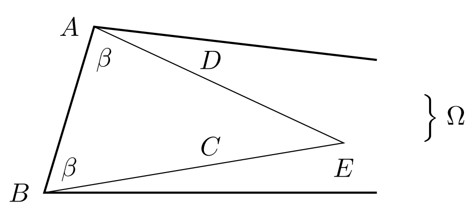
Theorem 9: If angles $BA\Omega$ and $B'A'\Omega'$ are equal and also angles $AB\Omega$ and $A'B'\Omega'$, then segments AB and $A'B'$ are equal and the figures are congruent. (AA criterion)

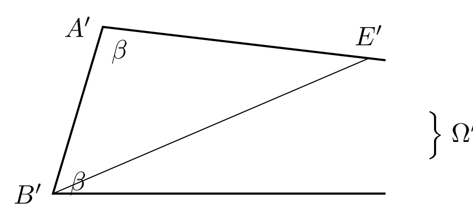




Statement	Reason
1. $\angle BA\Omega = \angle B'A'\Omega' = \alpha$, $\angle AB\Omega = \angle A'B'\Omega' = \beta$,	1. given
2. $AB \neq A'B'$, WLOG $AB > A'B'$	2. added premise
3. $AC = A'B'$	3. constructed (I.3)
4. $AC\Omega \cong A'B'\Omega'$	4. Theorem 8
5. $\angle AC\Omega = \angle A'B'\Omega'$	5. follows from (4)
6. exterior $\angle AC\Omega$ equals interior $\angle CB\Omega$ for $\triangleright BC\Omega$	6. both equal β
7. $AB = A'B'$	7. (6) contradicts Theorem 7
8. $\triangleright AB\Omega \cong \triangleright A'B'\Omega'$,	8. Theorem 8

Theorem 10: If segments AB and $A'B'$, angles $AB\Omega$ and $BA\Omega$, and angles $A'B'\Omega'$ and $B'A'\Omega'$, are equal, then all four angles are equal to one another and the figures are congruent.





Statement	Reason
1. $AB = A'B'$, $\angle AB\Omega = \angle BA\Omega = \alpha$, $\angle A'B'\Omega' = \angle B'A'\Omega' = \beta$	1. given
2. $\alpha \neq \beta$, WLOG $\alpha > \beta$	2. added premise
3. $\angle BAD = \beta = \angle ABC$	3. constructed (I.23)
4. BC intersects $A\Omega$ at T (not shown)	4. Theorem 1
5. AD intersects BC at E	5. Axiom of Pasch for $\triangle ABT$
6. choose E' on $A'\Omega'$ so that $A'E' = AE$, draw $B'E'$	6. constructed (I.3)
7. $\triangle EAB \cong \triangle E'A'B'$	7. SAS
8. $\angle ABE = \angle A'B'E'$	8. CPCTE
9. $\angle A'B'E' = \angle A'B'\Omega'$	9. both equal β
10. $\alpha = \beta$	10. (9) is a contradiction
11. $\triangleright AB\Omega \cong \triangleright A'B'\Omega'$	11. Theorem 8

Exercises

2. In the figure $AB\Omega$, the sum of angles $AB\Omega$ and $BA\Omega$ is always less than two right angles.
3. If a transversal meets two lines, making the sum of the interior angles on the same side equal to two right angles, then the two lines cannot meet and are not parallel, they are non-intersecting lines
4. Given two parallel lines, $A\Omega$ and $B\Omega$, and two other lines, $A'C'$ and $B'D'$, prove that if segments AB and $A'B'$, angles $BA\Omega$ and $B'A'C'$, and angles $AB\Omega$ and $A'B'D'$, are equal, then $A'C'$ and $B'D'$ are parallel.
5. If angles $AB\Omega$ and $BA\Omega$ are equal, the figure is in the nature of an isosceles triangle with vertex an ideal point. Prove that, if M is the midpoint of AB, $M\Omega$ is perpendicular to AB. Show also that the perpendicular to AB at M is parallel to $A\Omega$ and $B\Omega$ and that all points on it are equally distant from those two lines.
6. Prove that if, in figure $AB\Omega$, the perpendicular to AB at its midpoint is parallel to $A\Omega$ and $B\Omega$, then the angles at A and B are equal.
7. If, for two figures $AB\Omega$ and $A'B'\Omega'$, angles $AB\Omega$ and $A'B'\Omega'$ are equal but segment AB is greater than segment $A'B'$, then angle $BA\Omega$ is smaller than angle $B'A'\Omega'$.

41. The Angle of Parallelism.

From Theorem 8 of the preceding section, it is evident that the angle of parallelism $\Pi(h)$ for any given distance h is constant. Furthermore, as a consequence of Theorem 7, it follows that

$$0 < h_1 < h_2 \;\Rightarrow\; \Pi(h_1) > \Pi(h_2),$$

that is, Π is a strictly decreasing function.

We know, from the theorem of Section 37 [Theorem 1], that every distance has a corresponding angle of parallelism. It has just been pointed out that this angle is always the same for any given distance, that the angle increases as the distance decreases and decreases as the distance increases. Presently it will be shown that to every acute angle there corresponds a distance for which the angle is the angle of parallelism. In this case equal angles must, of course, have equal corresponding distances. Putting these results together, we conclude that

$$\lim_{\delta\to 0}\bigl(\Pi(h+\delta)-\Pi(h)\bigr)=0,$$

and consequently that $\Pi(h)$ varies continuously if h does. [The function Π is monotone and satisfies the intermediate value property and is thus continuous.]

It should perhaps be noted here that, so far, no particular unit has been specified for measuring either distances or angles. The functional relationship implied has been purely geometric. Later, when definite units have been adopted, the analytic form of $\Pi(h)$ [which turns out to be $2\arctan(e^{-h})$] will be obtained. However, as h approaches zero, $\Pi(h)$ approaches a right angle, and we may write $\Pi(0)=\pi/2$, where π is, for the present, used merely as a symbol to denote a straight angle. As h becomes infinite, $\Pi(h)$ approaches zero, or, in the conventional notation, $\Pi(\infty)=0$.

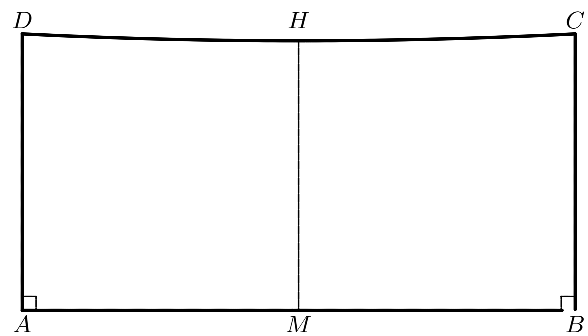
Moreover, there is no reason why we should not attach a meaning to $\Pi(h)$ for h negative. There is nothing which compels us to do this; we do it solely because it will prove convenient. Such a generalization will enable us to avoid certain exceptions later on. The definition of $\Pi(h)$, when h is negative, is a matter of choice, but we shall choose methodically.

As h, being positive, decreases, $\Pi(h)$ increases; when h is zero, $\Pi(h)$ is a right angle. If we think of h as continuing to decrease, becoming negative, we naturally choose to regard $\Pi(h)$ as continuing to increase and becoming obtuse. Briefly, $\Pi(-h)$ is defined by the relation $\Pi(h)+\Pi(-h)=\pi$.

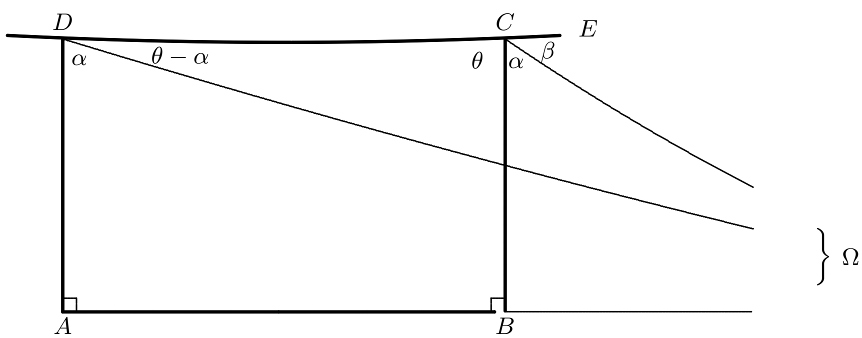
42. The Saccheri Quadrilateral.

It will be recalled that, as a basis for his investigations, Saccheri made systematic use of a quadrilateral formed by drawing equal perpendiculars at the ends of a line segment on the same side of it and connecting their extremities. This birectangular, isosceles quadrilateral is commonly called a *Saccheri Quadrilateral*. We shall study some of its properties. The side adjacent to the two right angles is known as the *base*, the opposite side as the *summit* and the angles adjacent to the summit as the *summit angles*.

Theorem 11: The line joining the midpoints of the base and summit of a Saccheri Quadrilateral is perpendicular to both of them; the summit angles are equal and acute.



Statement	Reason
1. $\square ABCD$ is a Saccheri Quadrilateral	1. given
2. M, H are the midpoints of AB, CD	2. constructed (I.10)
3. $\square DAMH \cong \square CBMH$	3. SASSS
4. $\angle AMH = \angle BMH$, $\angle DHM = \angle CHM$	4. CPCQE
5. MH is perpendicular to both AB and CD	5. follows from (5)
6. $\angle ADC = \angle BCD$	6. CPCQE



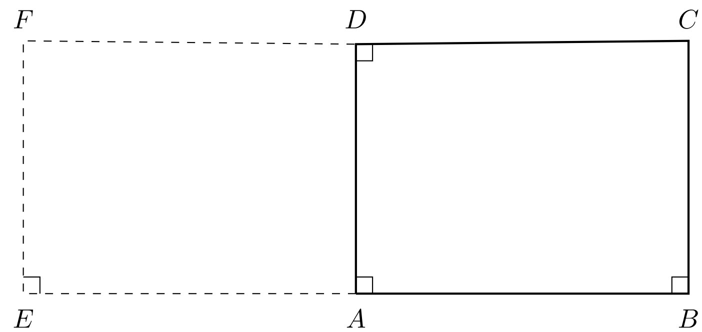
7. $D\Omega$, $C\Omega$ parallel toward the right to AB extended	7. assumed to exist
8. $\angle AD\Omega = \angle BC\Omega$ (labeled α)	8. equal angles of parallelism (Thm 8)
9. $\angle ADC = \angle BCD$ (labeled θ)	9. equal summit angles (from (6))
10. $\angle EC\Omega > \angle CD\Omega$ ($\beta > \theta - \alpha$ as labeled)	10. Theorem 7 for $\triangleright CD\Omega$
11. $\pi = \beta + \alpha + \theta > 2\theta$	11. $\angle DCE = \pi$, (10)
12. θ is an acute angle	12. simple consequence of (11)

Corollary 12: The base and summit of a Saccheri Quadrilateral are non-intersecting lines.

43. The Lambert Quadrilateral.

The reader will recollect that Lambert used, as a fundamental figure in his researches, a quadrilateral with three of its angles right angles. This trirectangular quadrilateral, which we shall call a *Lambert Quadrilateral*, has an important part to play in later developments.

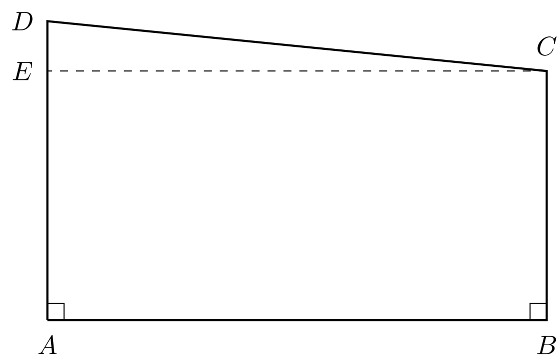
Theorem 13: In a Lambert Quadrilateral the fourth angle is acute.



Statement	Reason
1. $\square ABCD$ is a Lambert Quadrilateral with right angles indicated	1. given
2. extend BA to E so that $AE = BA$	2. constructed (I.3)
3. erect $EF \perp BE$ so that $EF = BC$	3. constructed (I.11, I.3)
4. join F to D	4. postulate
5. $\square DABC \cong \square DAEF$	5. SASAS
6. $\angle FDA$ is a right angle	6. CPCQE
7. FDC is a straight line	7. I.14
8. $\square FEBC$ is a Saccheri Quadrilateral	8. by definition
9. $\angle DCB$ is acute	9. Theorem 11

A useful theorem in regard to a more general quadrilateral, with only two right angles, may very well be inserted here. From it come immediately some important properties of the Saccheri and Lambert Quadrilaterals.

Theorem 14: If, in the quadrilateral $ABCD$, the angles at two consecutive vertices A and B are right angles, then the angle at C is larger than or smaller than the angle at D according as AD is larger than or smaller than BC, and conversely.



Statement	Reason
1. $\square ABCD$ with right angles at A and B	1. given
2. $\angle BCD = \angle C, \ \ \angle ADC = \angle D$	2. notation
3. $AD = BC \ \Rightarrow \ \angle C = \angle D$	3. $\square ABCD$ is a Saccheri Quadrilateral
4. $AD > BC$	4. added premise
5. choose E on AD so that $AE = BC$	5. constructed (I.3)
6. $\angle AEC = \angle BCE$	6. $\square ABCE$ is a Saccheri Quadrilateral
7. $\angle BCE = \angle AEC > \angle D$	7. I.16
8. $AD > BC \ \Rightarrow \ \angle C > \angle D$	8. statements (4)-(7)
9. $AD < BC \ \Rightarrow \ \angle C < \angle D$	9. similar reasoning
10. $\angle C = \angle D$	10. added premise
11. $AD \not> BC$, $AD \not< BC$	11. by (8) and (9)
12. $AD = BC$	12. law of trichotomy
13. $\angle C = \angle D \ \Rightarrow \ AD = BC$	13. statements (10)-(12)
14. $\angle C < \angle D \ \Rightarrow \ AD < BC$	14. similar reasoning
15. $\angle C > \angle D \ \Rightarrow \ AD > BC$	15. similar reasoning

Corollary 15: In a Lambert Quadrilateral, the sides adjacent to the acute angle are greater than their respective opposite sides.

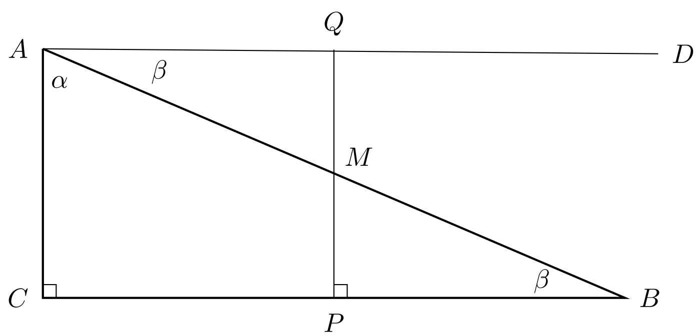
Corollary 16: The summit of a Saccheri Quadrilateral is larger than its base.

Exercises

8. Prove Corollary 12.
9. Prove Corollary 15.
10. Prove Corollary 16.
11. If, in quadrilateral $ABCD$, the angles at A and B are right angles and the angles at C and D are equal, prove that the figure is a Saccheri Quadrilateral.
12. Prove that the diagonals of a Saccheri Quadrilateral are equal but that these diagonals do not bisect each other.
13. Prove that, if perpendiculars are drawn from the extremities of one side of a triangle to the line passing through the midpoints of the other two sides, a Saccheri Quadrilateral is formed. As a consequence, prove that the perpendicular bisector of any side of a triangle is perpendicular to the line joining the midpoints of the other two sides.
14. Prove that the segment joining the midpoints of two sides of a triangle is less than one-half the third side.
15. Show that a line through the midpoint of one side of a triangle perpendicular to the line which bisects a second side at right angles bisects the third side.
16. Prove that the line joining the midpoints of the equal sides of a Saccheri Quadrilateral is perpendicular to the line joining the midpoints of the base and summit and that it bisects the diagonals.

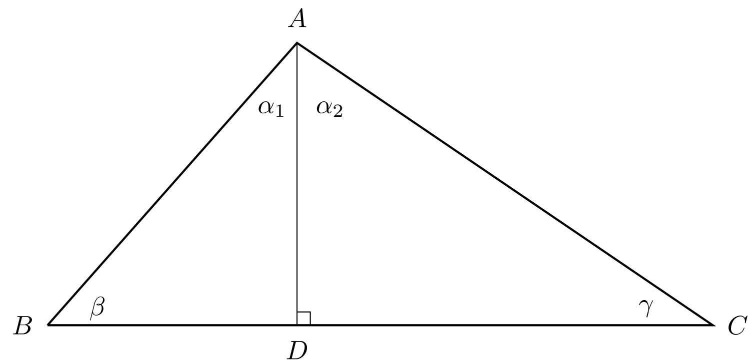
44. The Sum of the Angles of a Triangle.

Theorem 17: The sum of the angles of every right triangle is less than two right angles.



Statement	Reason
1. $\triangle ABC$ is a right triangle with $\angle C = \pi/2$	1. given
2. $\angle A = \alpha$, $\angle B = \beta$	2. notation
3. $\angle BAD = \angle B = \beta$	3. constructed (I.23)
4. M is midpoint of AB	4. constructed (I.10)
5. $MP \perp CB$	5. constructed (I.12)
6. P is on CB between C and B	6. Axiom of Pasch and I.28
7. $AQ = BP$, join M and Q	7. constructed (I.3)
8. $\triangle MBP \cong \triangle MAQ$	8. SAS
9. $\angle AMQ = \angle BMP$, $\angle MQA = \pi/2$	9. CPCTE
10. $\pi = \angle BMP + \angle AMP = \angle AMQ + \angle AMP$	10. AMB is a straight line, (9)
11. QMP is a straight line	11. I.14
12. $\square CPQA$ is a Lambert Quadrilateral	12. definition
13. $\alpha + \beta < \pi/2$	13. Theorem 13
14. $\angle A + \angle B + \angle C < \pi$	14. follows from (1) and (13)

Theorem 18: The sum of the angles of every triangle is less than two right angles.

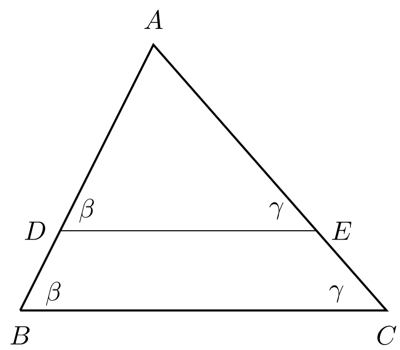


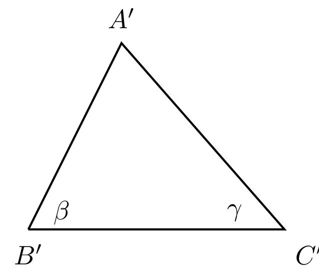
Statement	Reason
1. $\triangle ABC$ is a triangle, $\angle A = \alpha = \alpha_1 + \alpha_2$, $\angle B = \beta$, $\angle C = \gamma$	1. given, notation
2. WLOG angles β and γ are acute	2. I.17
3. $AD \perp BC$	3. constructed (I.12)
4. D is on BC between B and C	4. I.16
5. $\big(\alpha_1 + \beta + \pi/2\big) + \big(\alpha_2 + \gamma + \pi/2\big) < \pi + \pi$	5. Theorem 17
6. $\alpha + \beta + \gamma < \pi$	6. follows from (5)

The difference between two right angles and the angle-sum of a triangle is called the *defect* of the triangle.

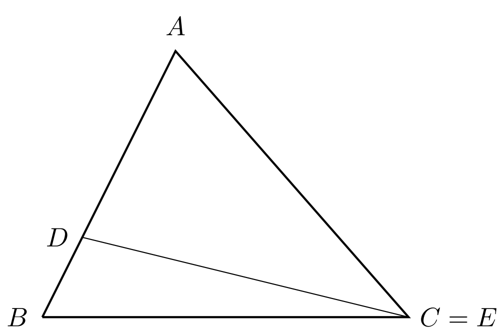
Corollary 19: The sum of the angles of every quadrilateral is less than four right angles.

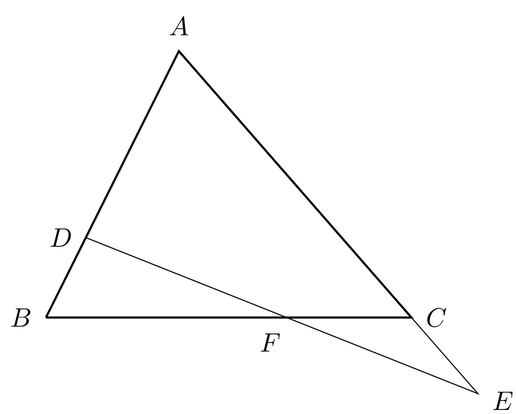
Theorem 20: If the three angles of one triangle are equal, respectively, to the three angles of a second, then the two triangles are congruent.





Statement	Reason
1. $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$,	1. given
2. $AB \neq A'B'$, WLOG $AB > A'B'$	2. added premise
3. $AD = A'B'$, $AE = A'C'$	3. constructed (I.3)
4. $AC > A'C'$ so E between A and C	4. see below (10–11)
5. $\triangle DAE \cong \triangle B'A'C'$	5. SAS
6. $\angle EDA = \angle B' = \angle B = \beta$, $\angle AED = \angle C' = \angle C = \gamma$,	6. CPCTE, hypothesis, notation
7. angle sum for quadrilateral $\square BCED$ is 2π	7. ADB and AEC are straight angles
8. $AB = A'B'$	8. (7) contradicts Corollary 19
9. $\triangle ABC \cong \triangle A'B'C'$	9. ASA





10. $AC = A'C' = AE \;\Rightarrow\; \angle ACD = \angle ACB$, a contradiction	10. D is between A and B
11. $AC < A'C' = AE \;\Rightarrow\; \angle CEF = \angle ACF$, a contradiction	11. I.16

Thus we reach the remarkable conclusion that in Hyperbolic Geometry similar triangles, or indeed similar polygons, of different sizes do not exist. We shall show in Section 54 how to construct a triangle, given the three angles.

Exercises

17. Prove that two Saccheri Quadrilaterals with equal summits and equal summit angles, or with equal bases and equal summit angles, are congruent.
18. A segment joining a vertex of a triangle to a point on the opposite side is called a Cevian. A Cevian divides a triangle into two subtriangles, and one or both of these can be subdivided by Cevians, and so on. Prove that, if a triangle is subdivided by arbitrary Cevians into a finite number of triangles, the defect of the triangle is equal to the sum of the defects of the triangles in the partition. What does this suggest about the defect of a triangle as compared to its size?