Total: Name:

Math 126 **Check-Up Quiz on Chapter 3** **Spring 2012**

Write neat, concise, and accurate solutions to each problem. No calculators are allowed.

1. Find the limit of the sequence $\left\{\frac{3n-1}{\sqrt{2n^2-n+4}}\right\}$.

Since $\lim_{n\to\infty} \frac{3n-1}{\sqrt{2n^2-n+4}} = \lim_{n\to\infty} \frac{3n-1}{\sqrt{2n^2-n+4}} \cdot \frac{1/n}{1/n}$

$= \lim_{n\to\infty} \frac{3-1/n}{\sqrt{2-\frac{1}{n}+4/n^2}} = \frac{3}{\sqrt{2}}$,

the limit of the sequence is $3/\sqrt{2}$.

2. Find the limit of the sequence $\left\{\left(\frac{2n+1}{2n}\right)^n\right\}$.

Note that $\left(\frac{2n+1}{2n}\right)^n = \left(1+\frac{1/2}{n}\right)^n$ for all n.

The limit is thus $e^{1/2}$ (by a theorem).

3. Find the limit of the sequence $\left\{4\sqrt[n]{7} - \sqrt[n]{n^2}\right\}$.

Since $\sqrt[n]{n^2} = \sqrt[n]{n}\cdot\sqrt[n]{n}$ and both $\{\sqrt[n]{7}\}$ and $\{\sqrt[n]{n}\}$ converge to 1, the limit of $\{4\sqrt[n]{7} - \sqrt[n]{n^2}\}$ is 3.

4. Give an example of a bounded sequence that does not converge. $\{(-1)^n\}$

5. Give an example of a decreasing sequence that does not converge. $\{-n^2\}$

6. Give an example of a convergent sequence that is not monotone. $\left\{\frac{(-1)^n}{n}\right\}$

be careful with notation

7. Give an example of a divergent series whose terms converge to 0. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

8. Find the sum of the series $8 - 6 + \frac{9}{2} - \frac{27}{8} + \cdots$.

This is a geometric series with $r = -\frac{3}{4}$ so the sum of the series is $\frac{8}{1-(-\frac{3}{4})} = \frac{32}{7}$.

9. Find the sum of the series $\sum_{k=1}^{\infty} \frac{3^{k-1}}{5^{2k}}$.

$$\sum_{k=1}^{\infty} \frac{3^{k-1}}{5^{2k}} = \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{3}{25}\right)^k = \frac{\frac{1}{3}\cdot\frac{3}{25}}{1-\frac{3}{25}} = \frac{1}{22}.$$

10. Consider the sequence $\{a_n\}$ defined by $a_1 = 1$ and $a_{n+1} = 7 - \frac{3}{a_n}$ for each $n \geq 1$. Use mathematical induction to prove that $\{a_n\}$ is an increasing sequence.

We will use the PMI to prove that $a_n < a_{n+1}$ for all positive integers n. Since $a_1 = 1$ and $a_2 = 7 - \frac{3}{1} = 4$, we see that $a_1 < a_2$. Now suppose that $a_k < a_{k+1}$ for some positive integer k. We then have

$$a_k < a_{k+1} \Rightarrow \frac{1}{a_k} > \frac{1}{a_{k+1}}$$
$$\Rightarrow -\frac{3}{a_k} < -\frac{3}{a_{k+1}}$$
$$\Rightarrow 7 - \frac{3}{a_k} < 7 - \frac{3}{a_{k+1}} \Rightarrow a_{k+1} < a_{k+2},$$

as desired. By the PMI we find that $a_n < a_{n+1}$ for all positive integers n. It follows that $\{a_n\}$ is an increasing sequence.

11. Determine whether or not the series $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ converges.

Note that $k > \ln k$ for all $k \geq 1$. It follows that $\frac{1}{\ln k} > \frac{1}{k}$ for all $k \geq 2$. Since the series $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, the series $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ diverges by the Comparison Test.

12. Determine whether or not the series $\sum_{k=1}^{\infty} \frac{2k-1}{k^3+5k-2}$ converges.

The terms of this series resemble $\frac{2}{k^2}$. Since

$$\alpha = \lim_{k\to\infty} \frac{\frac{2k-1}{k^3+5k-2}}{2/k^2} = \lim_{k\to\infty} \frac{2k^3-k^2}{2k^3+10k-4} = 1$$

and the series $\sum_{k=1}^{\infty} \frac{2}{k^2}$ converges, the series $\sum_{k=1}^{\infty} \frac{2k-1}{k^3+5k-2}$ converges by the Limit Comparison Test.

13. Classify the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k+1}$ as absolutely convergent, conditionally convergent, or divergent.

We first consider $\sum_{k=1}^{\infty} \frac{1}{4k+1}$. Since $\frac{1}{4k+1} \geq \frac{1}{5k}$ for all $k \geq 1$ and $\sum_{k=1}^{\infty} \frac{1}{5k}$ diverges, we see that $\sum_{k=1}^{\infty} \frac{1}{4k+1}$ diverges by the Comparison Test. However, the sequence $\left\{\frac{1}{4k+1}\right\}$ converges to 0 and is easily seen to be decreasing. Thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k+1}$ converges by the Alternating Series Test. We conclude that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k+1}$ is conditionally convergent.

14. Determine whether or not the series $\sum_{k=1}^{\infty} \frac{(2k)!}{7^k (k!)^2}$ converges.

We will use the Ratio Test.

$$l = \lim_{k\to\infty} \frac{(2k+2)!}{7^{k+1}((k+1)!)^2} \cdot \frac{7^k (k!)^2}{(2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{7(k+1)^2} = \frac{4}{7}$$

Since $l < 1$, the series converges.

15. Find a power series centered at 0 that represents the function $f(x) = \frac{x}{1-x^2}$.

Using the formula for a geometric series,

$$f(x) = \frac{x}{1-x^2} = \sum_{k=0}^{\infty} x(x^2)^k = \sum_{k=0}^{\infty} x^{2k+1}.$$

$a = x,\ r = x^2$

The power series for $f(x)$ is $\sum_{k=0}^{\infty} x^{2k+1}$.

16. Find the interval of convergence for the power series $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{3^k(5k+2)}(x-4)^k$.

Using the Ratio Test

$$l = \lim_{k\to\infty} \left| \frac{(-1)^{k+2}(x-4)^{k+1}}{3^{k+1}(5k+7)} \cdot \frac{3^k(5k+2)}{(-1)^{k+1}(x-4)^k} \right|$$

$$= \lim_{k\to\infty} \frac{|x-4|}{3} \cdot \frac{5k+2}{5k+7} = \frac{|x-4|}{3}$$

We need $|x-4| < 3$ or $1 < x < 7$.

at $x=1$, $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{3^k(5k+2)}(-3)^k = \sum_{k=0}^{\infty} \frac{-1}{5k+2}$, which diverges (CT)

at $x=7$, $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{5k+2}$, which converges (AST)

The interval of convergence is $(1, 7]$.