Miscellaneous problems for Math 126

1. Find a simple formula, one that does not involve summation notation, for $\sum_{k=1}^{n}(2k-1)(2k+1)$.

2. Find the area of the region bounded by the graph of $y = x^4/4$, the tangent line to this curve when $x = 2$, and the x-axis.

3. Find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}3^k}{4^{k-1}}$.

4. Use the definition of the integral to evaluate $\displaystyle\int_0^2 (x^3 - 4x)\,dx$.

5. Evaluate $\displaystyle\int_{-1}^2 (2x^2 - 5x + 3)\,dx$.

6. Find the limit of the sequence $\left\{\left(1 - \frac{2}{3n}\right)^n\right\}$.

7. Find the limit of the sequence $\left\{\sqrt[n]{4n^2 + n + 3}\right\}$.

8. Use a table of integrals to find $\displaystyle\int \sqrt{4 - e^x}\,dx$.

9. Find the volume of the solid that is generated when the region that lies above the x-axis and below the curve $y = \sqrt{\dfrac{14(3-x)}{(x+1)(7-x)}}$ on the interval $[0, 3]$ is revolved around the x-axis.

10. Prove carefully that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{k^2+1}$ is conditionally convergent.

11. Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt[k]{5}}$ converges.

12. Without evaluating them, explain which of the integrals $\displaystyle\int_0^1 \sin^4(\pi x)\,dx$ or $\displaystyle\int_0^1 \sin^5(\pi x)\,dx$ is larger.

13. Find the Taylor series for the function $f(x) = \sqrt{x}$ centered at $a = 9$.

14. Express the value of each of the following integrals as an infinite series.

    a) $\displaystyle\int_0^1 e^{-x^2}\,dx$ b) $\displaystyle\int_0^{0.5} \frac{\sin x}{x}\,dx$ c) $\displaystyle\int_0^1 \cos(x^2)\,dx$

15. Find the center of mass of the solid that is generated when the region below the curve $y = 4e^{-x/4}$ and above the x-axis on the interval $[0, \infty)$ is revolved around the x-axis.

16. Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{3k+1}\right)^k$ converges.

17. Find the derivative of the function F defined by $F(x) = \displaystyle\int_0^{x^2} t\sqrt{t^3+4}\,dt$.

18. Find the limit of the sequence $\left\{\sqrt{k^2 - 7k + 15} - k\right\}$

19. Evaluate $\displaystyle\int \frac{3x+1}{\sqrt{13-12x-x^2}}\,dx$.

20. Evaluate $\displaystyle\int_1^8 \frac{x+2}{\sqrt[3]{x}}\,dx$.

21. Suppose that $v(t) = 3t - t^3$ gives the velocity in meters per second of a particle at time t seconds. Find the distance traveled by the particle for the time interval $0 \le t \le 4$.

22. Use trigonometric substitution to derive $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left|u + \sqrt{a^2+u^2}\right| + C$.

23. Find the volume of the solid that is generated when the region that lies above the x-axis and below the curve $y = 4\sqrt{x}\,e^{-x/5}$ on the interval $[0,\infty)]$ is revolved around the x-axis.

24. Define a sequence $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = 4 - (1/x_n)$ for $n \ge 1$. Prove that $1 \le x_n \le 5$ for all n, then prove that $\{x_n\}$ is a decreasing sequence. Conclude that $\{x_n\}$ converges and find its limit.

25. Evaluate $\displaystyle\int_0^2 \frac{x}{4+x^2}\,dx$.

26. Evaluate $\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$.

27. Evaluate $\displaystyle\int_0^2 \frac{x}{\sqrt{4+x^2}}\,dx$.

28. Evaluate $\displaystyle\int_0^2 \frac{x}{\sqrt{4+x}}\,dx$.

29. Evaluate $\displaystyle\int \arctan x\,dx$.

30. Evaluate $\displaystyle\int_2^\infty \frac{6}{2x+5}\,dx$.

31. Evaluate $\displaystyle\int_0^\infty \frac{6+e^{2x}}{e^{3x}}\,dx$.

32. Find the area of the region bounded by the curves $x^2y = 90$ and $40x + y = 130$.

33. Suppose that the base of a solid is the part of the parabola $y = 8 - 0.5x^2$ that lies above the x-axis and that each cross section perpendicular to the y-axis is a semicircle. Find the volume of this solid.

34. Evaluate the limit: $\displaystyle\lim_{n\to\infty} \frac{3n^4+2n^2+1}{3^3+6^3+9^3+\cdots+(3n)^3}$.

35. Find the volume of the solid that is generated when the region that lies below the curve $y = \ln x$ and above the x-axis on the interval $[1,e]$ is revolved around the y-axis.

36. Evaluate the limit: $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(5+\frac{3i}{n}\right)^2 \frac{3}{n}$.

37. Find, with explanation, upper and lower bounds for the value of $\displaystyle\int_1^8 \frac{1}{4+3\sqrt[3]{x}}\,dx$.

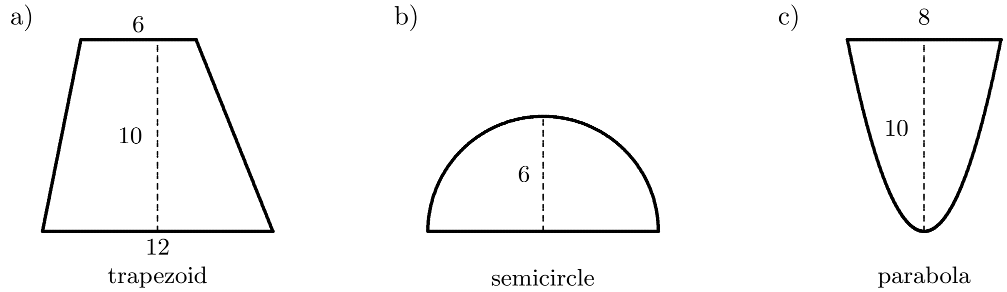
38. Find the volume of the solid that is generated when the portion of the parabola $y = (x-1)(3-x)$ that lies above the x-axis is revolved around the y-axis.

39. Prove the following statement: for each positive integer n, the integer $2^{5n-4} + 5^{2n-1}$ is divisible by 7.

40. Find the center of mass of the region that lies below the curve $y = x^4/10$ and above the x-axis on the interval $[0, 3]$.

41. Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{12}{3k+2}$ converges.

42. Find the force exerted by a liquid with weight density w on one side of each vertically submerged plate. The units on the figures are feet and the top of each plate is six feet beneath the surface of the liquid.



43. Use the trapezoid rule and Simpson's rule with $n = 6$ to approximate $\displaystyle\int_1^2 \sqrt{1+x^3}\,dx$ to four decimal places.

44. Find the center of mass of the solid that is generated when the region bounded by $y = 4 - x^2$ in the first quadrant is revolved around the x-axis.

45. Use an appropriate function to approximate $\displaystyle\int_1^2 \frac{1}{\sqrt{4x^6-1}}\,dx$. Is your estimate high or low?

46. Find the length of the curve $y = 4x^{3/2}$ on the interval $[0, 10]$.

47. Evaluate $\displaystyle\int \frac{3x+1}{\sqrt{12x - x^2}}\,dx$.

48. Evaluate $\displaystyle\int \frac{4x-7}{2x+1}\,dx$.

49. Derive the reduction formula for tangent.

50. Find the Maclaurin series for the function $f(x) = \dfrac{1}{5-2x}$ and determine its interval of convergence.

51. Find the area of the region bounded by the curves $y = 2\sqrt{x}$ and $y = x^3/16$.

52. Find the volume of the solid that is generated when the region bounded by the curves $y = 4x$ and $y = x^2$ is revolved around (a) the x-axis (b) the y-axis.

53. Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^2}{k^4+3k^2+10}$ is absolutely convergent.

54. Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{5k+4}$ converges.

55. Evaluate $\displaystyle\int_0^1 \left(2t - 3 + 2\sqrt{1-t^2}\right) dt$.

56. Use a table of integrals to find $\int_0^1 \frac{x^2}{\sqrt{1+4x^2}}\,dx$.

57. Use trigonometric substitution to derive $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{\sqrt{a^2-u^2}}{u} - \arcsin(u/a) + C$.

58. Find the area under the curve $y = 7 - |2x-3|$ and above the x-axis on the interval $[0,5]$.

59. Evaluate $\int_1^4 \frac{1}{3x-2}\,dx$.

60. Use trigonometric substitution to derive $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u + \sqrt{u^2-a^2}\right| + C$.

61. Evaluate $\int \frac{\sqrt{x^2+4}}{x^4}\,dx$.

62. Evaluate $\int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x^4}\,dx$.

63. Evaluate $\int \frac{4x-1}{x^2+2x-15}\,dx$.

64. Evaluate $\int \frac{x^2+2x+4}{x^3+x^2+x+1}\,dx$.

65. Evaluate $\int \frac{3x-7}{2x^2+7x-9}\,dx$.

66. Find the length of the curve $y = \frac{4}{5}\,x^{5/4}$ on the interval $[0,9]$.

67. Use the trapezoid rule and Simpson's rule with $n = 4$ to approximate $\int_0^1 e^{-x^2/2}\,dx$ to four decimal places.

68. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for each positive integer n.

69. Prove that $f_1 f_2 + f_2 f_3 + f_3 f_4 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$ for each positive integer n, where f_n represents the nth Fibonacci number.

70. Give an example of a sequence that converges to 5 but is not monotone.

71. Give an example of a monotone sequence of negative numbers that does not converge.

72. Give an example of a bounded sequence that does not converge.

73. Find the limit of the sequence $\left\{\frac{k}{\sqrt{3k^2+4k+1}}\right\}$.

74. Find an integral expression for a function f such that $f(\pi) = 0$ and $f'(x) = \cos^2(3x^2)$.

75. Find the limit of the sequence $\left\{k\left(\sqrt[k]{10} - 1\right)\right\}$.

76. Find the limit of the sequence $\left\{\frac{4^n + n^2}{2^{2n-3} + n^7}\right\}$.

77. Evaluate $\int \frac{12x}{(3x-1)^2}\,dx$.

78. Evaluate $\displaystyle\int_0^9 \frac{1}{1+\sqrt{x}}\,dx$.

79. Evaluate $\displaystyle\int x\sec^2 x\,dx$.

80. Evaluate $\displaystyle\int_0^3 (4x+2|x-1|)\,dx$.

81. Use the squeeze theorem to prove that the sequence $\{5^n/n!\}$ converges to 0.

82. Suppose that the following table represents the velocity of a particle moving in a straight line.

t	(sec)	0	1	2	3	4	5	6
v	(m/sec)	0	5	12	15	14	6	0

Use Simpson's rule to approximate the distance traveled by the particle.

83. For each positive integer n, let $x_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3} + \cdots + \dfrac{1}{2n}$. Prove that the sequence $\{x_n\}$ converges.

84. Evaluate $\displaystyle\int \left(2\sqrt{x}+1\right)^2 dx$.

85. Evaluate $\displaystyle\int \frac{3x}{(2x^2+5)^3}\,dx$.

86. Let $x_1 = 5$ and let $x_{n+1} = \sqrt{5+x_n}$ for $n \geq 1$. Suppose you know that the sequence $\{x_n\}$ converges. Find its limit.

87. Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{2}{\sqrt[k]{10}}$ converges.

88. Find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k+5^k}{7^k}$.

89. Let $\sum_{k=1}^{\infty} a_k$ be an infinite series and suppose that $s_n = \dfrac{n+1}{1-3n}$ for all $n \geq 1$, where $\{s_n\}$ is its corresponding sequence of partial sums. Find a_1, a_2, a_{10}, and the sum of the series.

90. Use the Integral Test to show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges.

91. For each positive integer n, let $a_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} - \displaystyle\int_1^n \frac{dx}{x}$. Prove that $\{a_n\}$ is a decreasing sequence.

92. Find the center of mass of the region bounded by the curves $y = 2\sqrt{x}$ and $y = x^3/16$.

93. Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{4k-1}{k^2+5k+2}$ converges.

94. Find a simple formula, one that does not involve summation notation, for $\displaystyle\sum_{i=1}^{2n-1} \left((i+1)^7 - i^7\right)$.

95. Define a sequence $\{c_n\}$ by $c_1 = 2$ and $c_n = 1/(3-c_{n-1})$ for $n \geq 2$. Prove that $0 < c_n \leq 2$ for all positive integers n and that $c_{n+1} < c_n$ for all positive integers n. Conclude that $\{c_n\}$ converges and find its limit.

96. Determine whether or not the series $\sum_{k=1}^{\infty} \frac{2k^2+3}{k^4+7k-1}$ converges.

97. Determine whether or not the series $\sum_{k=1}^{\infty} \frac{5^k}{2^k+6^k}$ converges.

98. Determine whether or not the series $\sum_{k=1}^{\infty} \frac{3^k \sin k}{4^k}$ is absolutely convergent.

99. Evaluate $\lim_{x\to 0} \frac{1}{x^3} \int_0^x \left(1-e^{-2t^2}\right) dt$.

100. Determine whether or not the series $\sum_{k=1}^{\infty} \frac{(-3)^k k!}{3\cdot 7\cdot 11\cdot \cdots \cdot (4k-1)}$ converges.

101. Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{1}{k^2 3^k}(x-4)^k$.

102. Find the interval of convergence of the power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)2^k}(x-1)^k$. endpoints.)

103. Give an example of a power series with $[4,10)$ as its interval of convergence.

104. Find (in more familiar terms) the function represented by the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}(x+1)^k$.

105. Evaluate $\int \frac{3x+8}{x^2+4x+6}\, dx$.

106. Evaluate $\int_0^2 \left(e^{x/2}-x\right) dx$.

107. By differentiating an appropriate power series (see problem 3.11.2), find the sum of the series $\sum_{k=1}^{\infty} k^3 x^k$.

108. Use known series to find the Maclaurin series for the given function.

a) $f(x) = e^{-x/3}$ b) $g(x) = \sin(x^2)$ c) $h(x) = \frac{1-\cos x}{x}$

109. Use known Maclaurin series to determine in more familiar terms the given function.

a) $\sum_{k=0}^{\infty} \frac{1}{2^k k!}\, x^k$ b) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k}$ c) $\sum_{k=0}^{\infty} \frac{(-9)^k}{(2k)!}\, x^{2k+1}$

110. Find the Maclaurin series for the function $f(x) = 1/\sqrt{1-x}$.

111. Find the Taylor series for the function $f(x) = 1/(2x-1)$ centered at $a=6$. with $\rho = 5.5$.

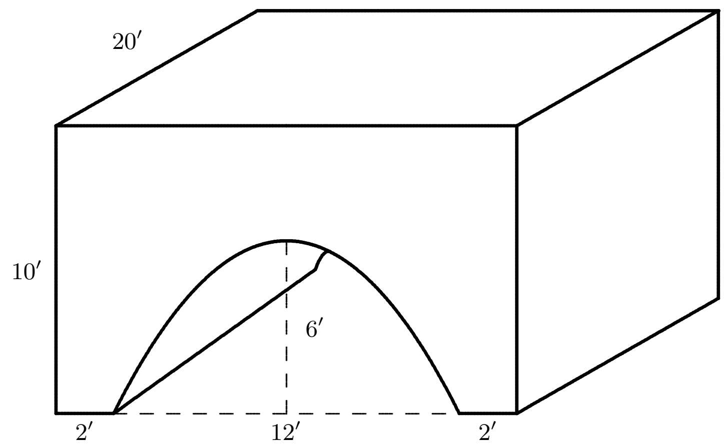
112. Evaluate $\int x\, e^{-x/2}\, dx$.

113. Evaluate $\int_1^{\infty} \frac{8}{(2x+5)^3}\, dx$.

The following problems might require a little more effort.

114. Consider two different solids. The base of each solid is a triangle with vertices $(0,0)$, $(2,4)$, and $(6,0)$. For solid A, each cross-section perpendicular to the y-axis is an equilateral triangle. For solid B, each cross-section perpendicular to the x-axis is a square. Find the ratio of the volume of solid A to the volume of solid B.

115. Find the number of cubic yards of concrete necessary for the culvert shown below. Assume that the arch of the culvert (which is empty space) has a parabolic shape.



116. Let $a_1 = 2$ and $a_{n+1} = 3 - (1/a_n)$ for each positive integer $n \geq 1$. Prove that $a_n = \dfrac{f_{2n+1}}{f_{2n-1}}$ for each positive integer n. Here f_n refers to the nth Fibonacci number.

117. For each positive integer n, let

$$y_n = \frac{1}{3n+2} + \frac{1}{3n+4} + \frac{1}{3n+6} + \cdots + \frac{1}{5n}.$$

Find the limit of the sequence $\{y_n\}$. (Try writing y_n in summation notation and think about integrals.)

118. Let $\{d_k\}$ be the sequence of positive integers in increasing order that do not have a 0 in their decimal representation. (For instance, 125 is in the sequence but 105 is not.) Prove that $\sum_{k=1}^{\infty} 1/d_k$ converges and has a sum less than 90.