A set E of real numbers has measure zero if for each $\epsilon > 0$ there exists a sequence (finite or infinite) $\{I_k\}$ of open intervals such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \ell(I_k) < \epsilon$. A property is said to hold almost everywhere on $[a,b]$ if it holds at each point of $[a,b] \setminus E$, where E is a set of measure zero.

Exercise 49: Prove that a countable union of sets of measure zero has measure zero.

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and let $c \in (a,b)$. The oscillation of the function f at the point c is defined by $\omega(f,c) = \lim_{r \to 0^+} \omega(f,[c-r,c+r])$.

Exercise 53: Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and let $c \in (a,b)$. Prove that the function f is continuous at c if and only if $\omega(f,c) = 0$.

Exercise 54/55: Suppose that the function f is Riemann integrable on $[a,b]$. Prove that f is continuous almost everywhere on $[a,b]$.

Solution: Let D be the set of all discontinuities of f that belong to the open interval (a,b). Since $D \cup \{a,b\}$ has measure zero if D does, it is sufficient to prove that D has measure zero. For each positive integer n, let $D_n = \{x \in (a,b) : \omega(f,x) \geq 1/n\}$. Since $D = \bigcup_{n=1}^{\infty} D_n$ by Exercise 53, it is sufficient to prove that each set D_n has measure zero (see Exercise 49). Fix $n \in \mathbb{Z}^+$ and let $\epsilon > 0$. By Theorem 5.10, there exists a partition $\{x_i : 0 \leq i \leq p\}$ of $[a,b]$ such that

$$\sum_{i=1}^{p} \omega(f,[x_{i-1},x_i])(x_i - x_{i-1}) < \frac{\epsilon}{2n}.$$

Let $S_0 = \{i : D_n \cap (x_{i-1},x_i) \neq \emptyset\}$ and note that $\omega(f,[x_{i-1},x_i]) \geq 1/n$ for each $i \in S_0$. It then follows that

$$\sum_{i \in S_0} (x_i - x_{i-1}) = n \sum_{i \in S_0} \frac{1}{n}(x_i - x_{i-1}) \leq n \sum_{i \in S_0} \omega(f,[x_{i-1},x_i])(x_i - x_{i-1}) \leq n \sum_{i=1}^{p} \omega(f,[x_{i-1},x_i])(x_i - x_{i-1}) < \frac{\epsilon}{2}.$$

We thus see that

$$D_n \subseteq \bigcup_{i \in S_0} (x_{i-1},x_i) \ \cup \ \bigcup_{i=1}^{p-1} \Big(x_i - \frac{\epsilon}{4p}, x_i + \frac{\epsilon}{4p}\Big)$$

and the sum of the lengths of the intervals comprising this union is

$$\sum_{i \in S_0} (x_i - x_{i-1}) + \sum_{i=1}^{p-1} \frac{\epsilon}{2p} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that D_n has measure zero. We conclude that f is continuous almost everywhere on $[a,b]$. ■