Some notes for Section 4.5

One of the most useful and remarkable results in the theory of complex integration is the Cauchy Integral Formula. This result shows that, under appropriate hypotheses, the values of an analytic function inside a simple closed contour are determined by the values of the function on the contour. The proof of this result is not difficult; it requires the Deformation Theorem and the fact that $\displaystyle\int_C \frac{ds}{s-z}\,ds = 2\pi i$ when C is any circle centered at z and traversed once in the positive direction.

Theorem 1: (Cauchy Integral Formula) If f is analytic within and on a positively oriented simple closed contour C, then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\,ds$$

for all points z in the interior of C.

Proof: Let z be a point in the interior of C. Since the interior of C is an open set, there exists $d > 0$ for which the set $\{s : |s-z| < d\}$ (a little disk) is contained within the interior of C. Let $\epsilon > 0$. Since f is continuous at z, there exists a positive number $\delta < d$ such that $|f(s) - f(z)| < \epsilon$ whenever $|s-z| \le \delta$. Let C_δ be the positively oriented circular path $\{s : |s-z| = \delta\}$. Note that C_δ is inside the contour C and that $|f(s) - f(z)| < \epsilon$ for all $s \in C_\delta$. By the Deformation Theorem (Theorem 8 in Section 4.4), we find that

$$\int_C \frac{f(s)}{s-z}\,ds = \int_{C_\delta} \frac{f(s)}{s-z}\,ds.$$

Using the ML inequality (Theorem 5 in Section 4.2), it follows that

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\,ds - f(z)\right| &= \left|\frac{1}{2\pi i}\int_{C_\delta} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{C_\delta} \frac{f(z)}{s-z}\,ds\right| \\ &= \frac{1}{2\pi}\left|\int_{C_\delta} \frac{f(s)-f(z)}{s-z}\,ds\right| \\ &\le \frac{1}{2\pi}\cdot\frac{\epsilon}{\delta}\cdot 2\pi\delta = \epsilon.\end{aligned}$$

Since this inequality holds for every $\epsilon > 0$, we conclude that $f(z) = \displaystyle\frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\,ds$. ■

As will be evident shortly, the Cauchy Integral Formula is an important tool in the proofs of a number of interesting results in complex analysis. However, this formula also makes it easy to evaluate some contour integrals when the contour is simple and closed. As an illustration, consider

$$\int_C \frac{\cos z}{z^2+1}\,dz,$$

where C is the positively oriented circle $\{z : |z-2i| = 2\}$. Since the function $\cos z/(z+i)$ is analytic within and on C and i is in the interior of C, we find that

$$\int_C \frac{\cos z}{z^2+1}\,dz = \int_C \frac{\cos z/(z+i)}{z-i}\,dz = 2\pi i\cdot\left.\frac{\cos z}{z+i}\right|_{z=i} = \pi\cos i = \pi\cosh 1.$$

This integral would be more difficult to evaluate using a parametrization of C.

The Cauchy Integral Formula shows that all of the values of an analytic function f in the interior of a simple closed contour are completely determined by the values of f on the contour. Take a moment to ponder the remarkable nature of this fact: the values on the boundary completely determine the values inside.

Proceeding formally, differentiating both sides of the Cauchy Integral Formula with respect to z yields

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds; \qquad f'''(z) = \frac{6}{2\pi i}\int_C \frac{f(s)}{(s-z)^4}\,ds;$$

$$f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}\,ds; \qquad f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds;$$

where the nth derivative follows by noting the appearance of a pattern. It turns out that this general formula is indeed correct. It shows that an analytic function is actually infinitely differentiable at each point in its domain of analyticity. This is an amazing result and certainly not true of real-valued functions. For example, the function $f(x) = x^{11/3}$ is differentiable for all values of x but $f^{(4)}(0)$ does not exist. Before proving this formula for derivatives, we offer two examples.

Problem: Evaluate $\int_\Gamma \frac{\cos(2z)}{(z-1)^4}\,dz$, where Γ is the positively oriented rectangle with vertices $-i$, $3-i$, $3+i$, and i.

Solution: Since $\cos(2z)$ is an entire function and 1 lies in the interior of Γ, we find that

$$\int_\Gamma \frac{\cos(2z)}{(z-1)^4}\,dz = \frac{2\pi i}{3!}\,f^{(3)}(1), \quad \text{where } f(z) = \cos(2z).$$

Since it is easy to see that $f^{(3)}(z) = 8\sin(2z)$, we find that

$$\int_\Gamma \frac{\cos(2z)}{(z-1)^4}\,dz = \frac{2\pi i}{3!}\cdot 8\sin 2 = \frac{8\pi i}{3}\sin 2.$$

Problem: Solve Example 5 in the textbook using another method.

Solution: The textbook solution uses the Cauchy Integral Formula corresponding to the function and its derivative. Here is another way to view this result. Referring to the statement of the problem, we find that

$$\int_C \frac{2z+1}{z(z-1)^2}\,dz = \int_C \Big(\frac{A}{z} + \frac{B}{z-1} + \frac{D}{(z-1)^2}\Big)\,dz = 2\pi i(B-A).$$

You should pause to consider why this is true. [Note a positively oriented circle around 1 and a negatively oriented circle around 0.] Adding the partial fractions reveals that $A + B = 0$ since it represents the coefficient of z^2. Using our 'cover and plug' method, we find that $A = 1$ and thus $B = -1$. With this information, the value of the integral is easily found to be $-4\pi i$.

Theorem 2: (Cauchy Integral Formula for Derivatives) If f is analytic within and on a positively oriented simple closed contour C, then f has derivatives of all orders at each point within C and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}}\, ds$$

for all z in the interior of C and for all positive integers n.

Proof: Let M be the maximum value of the function f on the closed contour C (this value exists since f is continuous and C is a compact set) and let L be the length of C. We first prove that the values for f' satisfy the formula given in the theorem. Let z be a point inside C. Since the interior of C is an open set, there exists $d > 0$ for which the set $\{s : |s - z| < 2d\}$ is contained within the interior of C. Let $\epsilon > 0$. Choose a positive number $\delta < d$ for which $\delta < d^3\epsilon/(ML)$. Suppose that w is any complex number that satisfies $0 < |w - z| < \delta$. Note that

$$|s - w| = |(s - z) - (w - z)| \geq |s - z| - |w - z| \geq 2d - \delta > d$$

for all s on the contour C. The Cauchy integral formula yields

$$\begin{aligned} \frac{f(w) - f(z)}{w - z} &= \frac{1}{w - z} \cdot \frac{1}{2\pi i}\Big(\int_C \frac{f(s)}{s - w}\, ds - \int_C \frac{f(s)}{s - z}\, ds\Big) \\ &= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - w)(s - z)}\, ds. \end{aligned}$$

Using the ML inequality (Theorem 5 in Section 4.2) once again, it follows that

$$\begin{aligned} \Big|\frac{f(w) - f(z)}{w - z} - \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2}\, ds\Big| &= \frac{1}{2\pi}\Big|\int_C \Big(\frac{1}{(s - w)(s - z)} - \frac{1}{(s - z)^2}\Big) f(s)\, ds\Big| \\ &= \frac{|w - z|}{2\pi}\Big|\int_C \frac{f(s)}{(s - w)(s - z)^2}\, ds\Big| \\ &\leq \frac{|w - z|}{2\pi} \cdot \frac{ML}{d(2d)^2} \\ &< \frac{ML}{8\pi d^3}\,\delta < \frac{ML}{d^3}\,\delta < \epsilon. \end{aligned}$$

Referring to the limit definition of the derivative, we find that

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2}\, ds.$$

This establishes the result for $n = 1$.

From here, there are several options. We can proceed with induction; this works but the details do get rather messy. Another option is to prove that f' (as given above) is differentiable on and inside C. This is also messy, but not too bad. It then follows that f' is analytic on and inside C. (The 'on' part takes some work also.) Applying this result to f', we find that f'' is analytic on and inside C. Since this process can then be continued, it follows that f has derivatives of all orders at each point on and within C. To establish the formula for $f^{(n)}(z)$, we can use integration by parts (see Exercise 4.3.11, but this is no different than calculus). Since $f^{(n)}$ is analytic on and inside C, the Cauchy Integral Formula applied to $f^{(n)}$ gives

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_C \frac{f^{(n)}(s)}{s - z}\, ds$$

for all z inside C. With $u = (s-z)^{-1}$ and $dv = f^{(n)}(s)\,ds$, integration by parts yields

$$\frac{1}{2\pi i}\int_C \frac{f^{(n)}(s)}{s-z}\,ds = \frac{1}{2\pi i}\left((uv)\Big|_{z_I}^{z_T} - \int_C v\,du\right) = \frac{1}{2\pi i}\int_C \frac{f^{(n-1)}(s)}{(s-z)^2}\,ds$$

since the uv part is evaluated over a closed contour, thus giving 0. Using integration by parts a second time with $u = (s-z)^{-2}$ and $dv = f^{(n-1)}(s)\,ds$ yields

$$\frac{1}{2\pi i}\int_C \frac{f^{(n-1)}(s)}{(s-z)^2}\,ds = \frac{2}{2\pi i}\int_C \frac{f^{(n-2)}(s)}{(s-z)^3}\,ds.$$

The next two integration by parts steps would give

$$\frac{6}{2\pi i}\int_C \frac{f^{(n-3)}(s)}{(s-z)^4}\,ds \qquad \text{and} \qquad \frac{24}{2\pi i}\int_C \frac{f^{(n-4)}(s)}{(s-z)^5}\,ds.$$

From here, we can see that the kth step yields

$$\frac{1}{2\pi i}\int_C \frac{f^{(n)}(s)}{s-z}\,ds = \frac{k!}{2\pi i}\int_C \frac{f^{(n-k)}(s)}{(s-z)^{k+1}}\,ds.$$

When $k = n$, we have (since $f^{(0)} = f$)

$$\frac{1}{2\pi i}\int_C \frac{f^{(n)}(s)}{s-z}\,ds = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds,$$

as desired. ■

For the record, it is not necessary to understand each step in these proofs (it is good practice, but not essential). The important part for now is understanding the result and being able to use the result to solve problems.