Some notes for Chapter 5

Definition 1: Let $\{z_n\}$ be a sequence of complex numbers. The sequence $\{z_n\}$ converges to z if for each $\epsilon > 0$ there exists a positive integer N such that $|z_n - z| < \epsilon$ for all $n \geq N$. The sequence $\{z_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N such that $|z_m - z_n| < \epsilon$ for all $m, n \geq N$.

Suppose that $z_n = x_n + iy_n$ for each n and that $z = x + iy$. Noting that

$$|x_n - x| \leq |z_n - z|, \quad |y_n - y| \leq |z_n - z|, \quad |z_n - z| \leq |x_n - x| + |y_n - y|,$$

it follows that the sequence $\{z_n\}$ of complex numbers converges if and only if the sequences $\{x_n\}$ and $\{y_n\}$ consisting of the real and imaginary parts converge. Since Cauchy sequences of real numbers converge, we find that a sequence $\{z_n\}$ of complex numbers converges if and only if it is a Cauchy sequence.

Theorem 2: Let $\{z_n\}$ be a convergent sequence. Then the sequence $\{z_n\}$ is bounded and the sequence $\{z_{n+1} - z_n\}$ converges to 0.

Proof: Let z be the limit of the sequence. Corresponding to $\epsilon = 1$, there exists a positive integer N such that $|z_n - z| < 1$ for all $n > N$. For each $n > N$, we then have (using the triangle inequality)

$$|z_n| \leq |z_n - z| + |z| < 1 + |z|.$$

Letting $M = \max\{|z_1|, |z_2|, \ldots, |z_n|, 1 + |z|\}$, we find that $|z_n| \leq M$ for all n. This shows that $\{z_n\}$ is bounded. For the second part, let $\epsilon > 0$. Since $\{z_n\}$ converges to z, there exists a positive integer N such that $|z_n - z| < \epsilon/2$ for all $n \geq N$. It then follows that

$$|z_{n+1} - z_n| \leq |z_{n+1} - z| + |z - z_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n \geq N$. This shows that the sequence $\{z_{n+1} - z_n\}$ converges to 0. (We could also use linearity to conclude that the sequence converges to $z - z = 0$.) Note that the sequence $\{z_n - z_{n-1}\}$ also converges to 0, but it is not defined for $n = 1$. ∎

Definition 3: Given a sequence $\{a_k\}$, an infinite series is an expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots,$$

which represents the sum of all of the terms of the sequence $\{a_k\}$. For each positive integer n, let $s_n = \sum_{k=1}^{n} a_k$. The sequence $\{s_n\}$ is known as the sequence of partial sums of the series. A series converges if and only if its corresponding sequence of partial sums converges. The sum of a convergent series is the limit of its sequence of partial sums: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$.

A series $\sum_{k=1}^{\infty} a_k$ is really two sequences; the sequence of terms and the sequence of partials sums:

$$a_1, a_2, a_3, a_4, a_5, a_6, \ldots \qquad \text{and} \qquad a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots$$

It is important to realize the distinction between these two sequences.

Theorem 4: Let $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$ be two convergent series. Then

(a) $\lim\limits_{k\to\infty} a_k = 0$; (and thus a series $\sum\limits_{k=1}^{\infty} a_k$ diverges if $\lim\limits_{k\to\infty} a_k \neq 0$)

b) the series $\sum\limits_{k=1}^{\infty} \overline{a}_k$ converges and $\sum\limits_{k=1}^{\infty} \overline{a}_k = \overline{\sum\limits_{k=1}^{\infty} a_k}$;

a) the series $\sum\limits_{k=1}^{\infty} ca_k$ converges and $\sum\limits_{k=1}^{\infty} ca_k = c\sum\limits_{k=1}^{\infty} a_k$, where c is any complex number;

b) the series $\sum\limits_{k=1}^{\infty} (a_k + b_k)$ converges and $\sum\limits_{k=1}^{\infty} (a_k + b_k) = \sum\limits_{k=1}^{\infty} a_k + \sum\limits_{k=1}^{\infty} b_k$.

Proof: We prove part (a) only. Since $\sum\limits_{k=1}^{\infty} a_k$ converges, its corresponding sequence $\{s_n\}$ of partial sums converges to some number S. Referring to Theorem 2, we find that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (s_n - s_{n-1}) = S - S = 0$. ■

An important class of infinite series are the geometric series, which are series of the form

$$\sum_{k=0}^{\infty} c^k = 1 + c + c^2 + c^3 + c^4 + c^5 + \cdots,$$

where c is a complex number. To determine whether or not this series converges, we need to look at its sequence $\{s_n\}$ of partial sums. Some algebra yields (assuming that $c \neq 1$)

$$s_n - cs_n = \big(1 + c + c^2 + \cdots + c^n\big) - \big(c + c^2 + c^3 + \cdots + c^{n+1}\big) = 1 - c^{n+1} \quad \Rightarrow \quad s_n = \frac{1 - c^{n+1}}{1 - c}.$$

If $|c| < 1$, the sequence $\{c^{n+1}\}$ converges to 0 (since the sequence $\{|c|^n\}$ of real numbers converges to 0) and the sequence $\{s_n\}$ converges to $1/(1-c)$. If $|c| \geq 1$, then the series diverges since the terms do not go to 0 (refer to part (a) of Theorem 4). Let p be a positive integer and note that

$$\sum_{k=p}^{\infty} c^k = c^p + c^{p+1} + c^{p+2} + c^{p+3} + \cdots = c^p\big(1 + c + c^2 + c^3 + \cdots\big) = c^p \sum_{k=0}^{\infty} c^k = \frac{c^p}{1-c},$$

assuming that $|c| < 1$. It is important to keep this simple fact in mind when finding sums of series.

Example 5: Find the sum of the series $\displaystyle\sum_{k=1}^{\infty}\Big(\frac{1-i}{1+3i}\Big)^k$.

Solution: It is easy to verify that this is a geometric series with a c value that satisfies $|c| = 1/\sqrt{5}$. Since this value is less than 1, the series converges and

$$\sum_{k=1}^{\infty}\Big(\frac{1-i}{1+3i}\Big)^k = \frac{\dfrac{1-i}{1+3i}}{1 - \dfrac{1-i}{1+3i}} = \frac{1-i}{(1+3i)-(1-i)} = \frac{1-i}{4i} = -\frac{1}{4} - \frac{i}{4}.$$

Although the computations are a little more tedious when complex numbers are involved, these series behave much the same as for geometric series of real numbers.

Definition 6: A series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Theorem 7: If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges. [The converse is false!]

Proof: Let $\{s_n\}$ be the sequence of partial sums for the absolute value series and let $\{t_n\}$ be the sequence of partial sums for the other series. Suppose that $n > m$ and use the triangle inequality to obtain

$$|t_n - t_m| = \Big| \sum_{k=m+1}^{n} a_k \Big| \le \sum_{k=m+1}^{n} |a_k| = |s_n - s_m|.$$

Since the series $\sum_{k=1}^{\infty} |a_k|$ converges, the sequence $\{s_n\}$ is a Cauchy sequence. It then follows that the sequence $\{t_n\}$ is a Cauchy sequence and thus convergent. We conclude that the series $\sum_{k=1}^{\infty} a_k$ converges. ■

Since it is typically difficult to find a formula for the sequence of partial sums (the geometric series is a rare exception), it is usually not possible to directly obtain the sum of a series. You may recall some convergence tests from calculus. We will focus on just two of these.

Theorem 8: (Comparison Test) If $|a_k| \le b_k$ for all $k \ge K$ for some $K \in \mathbb{Z}^+$ and the series $\sum_{k=1}^{\infty} b_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges.

Proof: To simplify the proof, suppose that $K = 1$. With $s_n = \sum_{k=1}^{n} b_k$ and $t_n = \sum_{k=1}^{n} |a_k|$, we find that

$$|t_n - t_m| = \sum_{k=m+1}^{n} |a_k| \le \sum_{k=m+1}^{n} b_k = |s_n - s_m|.$$

As in the proof of Theorem 7, the series $\sum_{k=1}^{n} |a_k|$ converges. Then by Theorem 7, the series $\sum_{k=1}^{\infty} a_k$ converges. ■

Theorem 9: (Ratio Test) Let $\sum_{k=1}^{\infty} a_k$ be a series and suppose that $\ell = \lim_{k\to\infty} \big|a_{k+1}/a_k\big|$ exists. Then the series converges absolutely if $\ell < 1$ (and thus $\sum_{k=1}^{\infty} a_k$ converges) and diverges if $\ell > 1$.

Proof: Suppose $\ell < 1$. Let r be a number that satisfies $\ell < r < 1$. By the definition of a convergent sequence, there exists a positive integer p such that $\big|a_{k+1}/a_k\big| < r$ for all $k \ge p$. It follows that

$$|a_{p+1}| < r\,|a_p| = \left(\frac{|a_p|}{r^p}\right) r^{p+1}, \qquad |a_{p+2}| < r\,|a_{p+1}| < \left(\frac{|a_p|}{r^p}\right) r^{p+2},$$
$$|a_{p+3}| < r\,|a_{p+2}| < \left(\frac{|a_p|}{r^p}\right) r^{p+3}, \qquad |a_{p+4}| < r\,|a_{p+3}| < \left(\frac{|a_p|}{r^p}\right) r^{p+4},$$

and, in general, $|a_k| < \big(|a_p|/r^p\big) r^k$ for all $k > p$. Since the series $\sum_{k=1}^{\infty} \big(|a_p|/r^p\big) r^k$ is a convergent geometric series, the series $\sum_{k=1}^{\infty} |a_k|$ converges by the Comparison Test.

Now suppose that $\ell > 1$. By the definition of a convergent sequence, there exists a positive integer q such that $\big|a_{k+1}/a_k\big| > 1$ for all $k \ge q$. It follows (as above) that $|a_k| > |a_q| > 0$ for all $k > q$, which indicates that the sequence $\{a_k\}$ does not converge to 0. By Theorem 4, the series $\sum_{k=1}^{\infty} a_k$ diverges. ■

We now consider sequences and series of functions. As an example of each of these, consider the sequence

$$\left\{\frac{z}{1+kz}\right\}_{k=0}^{\infty} = z,\ \frac{z}{1+z},\ \frac{z}{1+2z},\ \frac{z}{1+3z},\ \frac{z}{1+4z},\ \ldots$$

and the series

$$\sum_{k=1}^{\infty} \frac{e^{kz}}{z^k} = \frac{e^z}{z} + \frac{e^{2z}}{z^2} + \frac{e^{3z}}{z^3} + \frac{e^{4z}}{z^4} + \frac{e^{5z}}{z^5} + \cdots.$$

When any given complex number is inserted for z, we obtain a sequence and a series of complex numbers, just like the type we have been studying. For example, when $z = 1$ for the sequence and $z = 2\pi i$ for the series, we obtain

$$\left\{\frac{z}{1+kz}\right\}_{k=0}^{\infty} = \left\{\frac{1}{1+k}\right\}_{k=0}^{\infty} = 1,\ \frac{1}{2},\ \frac{1}{3},\ \ldots \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{e^{kz}}{z^k} = \sum_{k=1}^{\infty} \frac{1}{(2\pi i)^k} = \frac{1}{2\pi i} + \frac{1}{(2\pi i)^2} + \frac{1}{(2\pi i)^3} + \cdots,$$

respectively. One of the key questions for sequences and series of functions is determining which values of z result in a sequence or series that converges. As a simple example, we know that the series $\sum\limits_{k=0}^{\infty} z^k$ converges for all values of z that satisfy $|z| < 1$; this is just the geometric series. For other series, the Ratio Test is often helpful.

Example 10: Find the values of z for which the series $\displaystyle\sum_{k=1}^{\infty} \frac{(z-1+i)^k}{k(1+2i)^k}$ converges.

Solution: Applying the Ratio Test, we find that

$$\begin{aligned}
\ell &= \lim_{k\to\infty} \left| \frac{(z-1+i)^{k+1}}{(k+1)(1+2i)^{k+1}} \div \frac{(z-1+i)^k}{k(1+2i)^k} \right| \\
&= \lim_{k\to\infty} \left| \frac{(z-1+i)^{k+1}}{(k+1)(1+2i)^{k+1}} \cdot \frac{k(1+2i)^k}{(z-1+i)^k} \right| \\
&= \lim_{k\to\infty} \frac{k}{k+1} \cdot \frac{|z-(1-i)|}{|1+2i|} \\
&= \frac{|z-(1-i)|}{\sqrt{5}}.
\end{aligned}$$

(In practice, we often omit the division step and go straight to the invert and multiply step.) It follows that the series converges for all z that satisfy $|z-(1-i)| < \sqrt{5}$ and diverges for all z that satisfy $|z-(1-i)| > \sqrt{5}$. The situation when $|z-(1-i)| = \sqrt{5}$ is more complicated and will (perhaps) be considered later in these notes. Observe that we have convergence of this series inside a circle of a certain center and radius.

By simple properties of derivatives, we know that

$$\frac{d}{dz}\big(f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z)\big) = f_1'(z) + f_2'(z) + f_3'(z) + \cdots + f_n'(z)$$

for differentiable functions; the derivative of a sum is the sum of the derivatives. This is true for any value of n. What about a formula such as

$$\frac{d}{dz}\sum_{k=1}^{\infty} f_k(z) = \sum_{k=1}^{\infty} f_k'(z), \qquad \left(\text{or similarly} \quad \int_\gamma \left(\sum_{k=1}^{\infty} f_k(z)\right) dz = \sum_{k=1}^{\infty} \int_\gamma f_k(z)\, dz \right)$$

where each f_k is a differentiable (integrable) function, which extends this result to series? It turns out that these results are only true in certain special cases. One way to guarantee these results involves uniform convergence; see the textbook for its definition. We are not going to worry too much about this concept and, furthermore, we will be working with series of functions for which these seemingly natural formulas do indeed hold.

Definition 11: If f is analytic at z_0, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \frac{f'''(z_0)}{6}(z-z_0)^3 + \cdots$$

is called the Taylor series of f at z_0. In the special (and often occurring) case in which $z_0 = 0$, the series is known as the Maclaurin series of f.

There exist functions that are infinitely differentiable at a point z_0 (so a Taylor series can be written down), but for which the function and the Taylor series are not equal. Fortunately, as shown by the next theorem, this is not the case for analytic functions.

Theorem 12: If f is analytic in the disk $\{z : |z - z_0| < r\}$, then the Taylor series for f at z_0 converges to $f(z)$ for all z in the disk. (Note that r can be taken to be the distance from z_0 to the nearest point at which the function f is not analytic.)

Proof: To simplify the notation, we assume that $z_0 = 0$. Suppose that $|z| < r$, choose a number ρ such that $|z| < \rho < r$, and let C be the positively oriented circle $|w| = \rho$. Without being too concerned with the technical details, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\,ds && \text{Cauchy Integral Formula} \\
&= \frac{1}{2\pi i}\int_C \frac{f(s)}{s}\cdot\frac{1}{1-(z/s)}\,ds && \text{algebra} \\
&= \frac{1}{2\pi i}\int_C \frac{f(s)}{s}\sum_{k=0}^{\infty}\Big(\frac{z}{s}\Big)^k\,ds && \text{geometric series } (|z/s| < 1) \\
&= \frac{1}{2\pi i}\int_C \sum_{k=0}^{\infty}\frac{f(s)z^k}{s^{k+1}}\,ds && \text{algebra} \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_C \Big(\frac{f(s)}{s^{k+1}}\,ds\Big)z^k && \text{uniform convergence} \\
&= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k && \text{Cauchy Integral Formula}
\end{aligned}$$

Hence, the Taylor series for f at 0 converges to $f(z)$ for all z that satisfy $|z| < r$. ■

The Taylor series for a function looks like a polynomial; the “only” difference is that the number of terms is infinite. Fortunately, these Taylor series behave in much the same way as polynomials. For example, Taylor series can be added (this is easy) and multiplied (but the distributive property gives a real mess). The only way for two Taylor series centered at z_0 to be equal is for the coefficients to all be equal. These facts make Taylor series useful for solving certain algebraic and differential equations. Examples will appear later.

One way to find the Taylor series for an analytic function f involves finding a pattern for all of its derivatives. For many functions, the higher order derivatives become rather messy. For example, consider taking ten derivatives of the function $\tan z$. However, the Maclaurin series for e^z is very easy to find; all the derivatives are the same and they all equal 1 at 0. Once we know the Maclaurin series for e^z, we can easily obtain the Maclaurin series for functions related to e^z.

For later use, we record some standard Maclaurin series.

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!}\, z^k = 1 + z + \frac{1}{2}\, z^2 + \frac{1}{6}\, z^3 + \frac{1}{24}\, z^4 + \cdots;$$

$$\sinh z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\, z^{2k+1} = z + \frac{1}{6}\, z^3 + \frac{1}{120}\, z^5 + \frac{1}{7!}\, z^7 + \cdots;$$

$$\cosh z = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\, z^{2k} = 1 + \frac{1}{2}\, z^2 + \frac{1}{24}\, z^4 + \frac{1}{6!}\, z^6 + \cdots;$$

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, z^{2k+1} = z - \frac{1}{6}\, z^3 + \frac{1}{120}\, z^5 - \frac{1}{7!}\, z^7 + \cdots;$$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, z^{2k} = 1 - \frac{1}{2}\, z^2 + \frac{1}{24}\, z^4 - \frac{1}{6!}\, z^6 + \cdots.$$

We mentioned above how easy it is to find the Maclaurin series for e^z. Once we have this series, we can use the following equations

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \quad \cosh z = \frac{d}{dz}\sinh z, \quad \sin z = \frac{\sinh(iz)}{i}, \quad \cos z = \frac{d}{dz}\sin z,$$

to easily determine the other Maclaurin series. We can perform further manipulations to find other Maclaurin series. Here are two examples.

$$\frac{\sin z - z}{z^2} = \frac{1}{z^2}\Big(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, z^{2k+1} - z\Big) = \frac{1}{z^2}\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\, z^{2k+1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\, z^{2k-1};$$

$$\begin{aligned}\int_0^z \frac{e^{(1+i)w} - 1}{w}\, dw &= \int_0^z \frac{1}{w}\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\left((1+i)w\right)^k - 1\Big)\, dw = \int_0^z \sum_{k=1}^{\infty} \frac{(1+i)^k}{k!}\, w^{k-1}\, dw \\ &= \sum_{k=1}^{\infty} \int_0^z \frac{(1+i)^k}{k!}\, w^{k-1}\, dw = \sum_{k=1}^{\infty} \frac{(1+i)^k}{k\, k!}\, z^k.\end{aligned}$$

Note that the path for the integral does not matter since the integrand is an entire function.

To find the Taylor series for $f(z) = \operatorname{Log} z$ centered at $z = 1$, we take a few derivatives and quickly note a pattern:

$$f'(z) = \frac{1}{z}, \quad f''(z) = \frac{-1}{z^2}, \quad f'''(z) = \frac{2}{z^3}, \quad f^{(4)}(z) = \frac{-6}{z^4}, \quad \ldots, \quad f^{(k)}(z) = (-1)^{k+1}\frac{(k-1)!}{z^k} \ \text{ for } k \geq 1.$$

Using the formula for the Taylor series (noting that $f(1) = 0$), we find that

$$\operatorname{Log} z = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, (z-1)^k.$$

This series converges for all z that satisfy $|z - 1| < 1$; notice that 1 is the distance from the center 1 to the nearest "bad" point, namely $z = 0$.

Definition 13: Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of complex numbers and let z_0 be a complex number. A power series centered at z_0 is an expression of the form

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + a_4(z-z_0)^4 + \cdots .$$

The constants a_k are known as the coefficients of the power series and the number z_0 is called the center of the power series.

A power series represents a function of z; for each fixed value of z, a power series becomes an infinite series and the sum of the series is the value of the function at z. The Ratio Test can often be used to determine the domain of a function defined as a power series. As we will see, power series are just the Taylor series for some analytic function. In fact, if we let $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ and treat f just like a polynomial, we quickly find that $f^{(k)}(z_0) = k!a_k$. (You should write out a few terms to make sure that you see this.) This shows that the coefficients a_k are precisely those for the Taylor series of f at z_0. To be more rigorous, we need uniform convergence and the Cauchy integral formula; see the textbook for some details.

Theorem 14: Given a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, one of the following occurs:

a) for some $\rho > 0$, the series converges if $|z-z_0| < \rho$ and diverges if $|z-z_0| > \rho$;

b) the series converges for all complex numbers;

c) the series converges only for $z = z_0$. ■

The number ρ is called the radius of convergence of the power series; this number is assumed to be ∞ in part (b) and to be 0 in part (c). In calculus, the set of points where a power series converges is an interval, but in the present context, the set of points where the series converges is the interior of a disk. Hence, the term 'radius of convergence' makes much more sense. You may also recall working with the interval of convergence. In calculus, this involved checking the two endpoints of the interval. In complex analysis, we have an entire disk of 'endpoints' so checking convergence on the boundary is requires more effort. The radius of convergence of a power series can usually be found using the Ratio Test. (Technically, we need a stronger test known as the Root Test, but we will not be concerned with this test).

Here is one way to think of the radius of convergence. We always have convergence at the center since all of the terms of the power series after the first term are 0 when evaluated at the center. As we start to move away from the center, the modulus $|z-z_0|$ increases. As these numbers get larger, it is "harder" for the series to converge. As we continue moving away from the center, we may eventually reach a point where the numbers are just too big and we get divergence. Once this happens, we will get divergence the "rest of the way out." However, in some cases, the moduli of the coefficients a_k go to 0 very quickly and this can provide a buffer for the large $|z-z_0|$ values. When this happens, the power series converges for all values of z. The Maclaurin series for e^z and $\sin z$ provide examples of this.

Example 15: Find the radius of convergence for the power series $\displaystyle\sum_{k=0}^{\infty} \frac{7k+4}{(2+2i)^k}\,(z-4i)^{3k}$.

Solution: Writing out the first few terms of this power series, we have

$$4 + \frac{11}{2+2i}\,(z-4i)^3 + \frac{18}{(2+2i)^2}\,(z-4i)^6 + \frac{25}{(2+2i)^3}\,(z-4i)^9 + \cdots.$$

Technically, we cannot apply the Ratio Test (certainly not in the form given in Exercise 5.3.2). The reason for this is that many of the coefficients are 0 making the ratio $|a_{k+1}/a_k|$ undefined. (For instance, it is clear that $0 = a_1 = a_2 = a_4$ and so on.) If you are inclined, you can use the Root Test since it works in this case but you do need to be somewhat comfortable with the concept of the limit superior of a sequence to apply this test directly. Since we are not assuming this knowledge, we use the Ratio Test; we just need to bring in the z values as well. By the Ratio Test, we consider the limit

$$\begin{aligned}
\ell &= \lim_{k\to\infty} \left| \frac{(7k+11)(z-4i)^{3k+3}}{(2+2i)^{k+1}} \cdot \frac{(2+2i)^k}{(7k+4)(z-4i)^{3k}} \right| \\
&= \lim_{k\to\infty} \frac{7k+11}{7k+4} \cdot \frac{|z-4i|^3}{|2+2i|} \\
&= \frac{|z-4i|^3}{\sqrt{8}}.
\end{aligned}$$

The series converges when $\ell < 1$; this occurs when $|z-4i|^3 < \sqrt{8}$, which is equivalent to $|z-4i| < \sqrt{2}$. The radius of convergence of the power series is $\sqrt{2}$.

Determining solutions to differential equations is one of the advantages of power series. We will not be doing a lot of this, but here is one example.

Example 16: Find an analytic function f such that $f''(z) = zf(z)$, $f(0) = 1$, and $f'(0) = 0$.

Solution: We assume that $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Writing out some terms, we see that

$$\begin{aligned}
f(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots; \\
zf(z) &= a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + a_4 z^5 + a_5 z^6 + \cdots = \sum_{k=3}^{\infty} a_{k-3} z^{k-2}; \\
f''(z) &= 2a_2 + 6a_3 z + 12a_4 z^2 + 20a_5 z^3 + 30a_6 z^4 + \cdots = \sum_{k=2}^{\infty} k(k-1)a_k z^{k-2}.
\end{aligned}$$

The conditions $f(0) = 1$ and $f'(0) = 0$ indicate that $a_0 = 1$ and $a_1 = 0$. In order for $f''(z) = zf(z)$, we need all of the coefficients to match up. (This is a consequence of the uniqueness of power series.) We thus have

$$2a_2 = 0, \quad 6a_3 = a_0, \quad 12a_4 = a_1, \quad 20a_5 = a_2, \quad 30a_6 = a_3, \quad \ldots; \qquad k(k-1)a_k = a_{k-3}.$$

The last equation indicates the pattern that appears for all $k \geq 3$. This pattern is also clear from the series representations given above. We thus find that

$$0 = a_1 = a_4 = a_7 = \cdots; \quad 0 = a_2 = a_5 = a_8 = \cdots; \quad a_3 = \frac{a_0}{3\cdot 2} = \frac{1}{3!}, \quad a_6 = \frac{a_3}{6\cdot 5} = \frac{1\cdot 4}{6!}, \quad a_9 = \frac{a_6}{9\cdot 8} = \frac{1\cdot 4\cdot 7}{9!}$$

and so on. Identifying the pattern for the a_{3k} coefficients, we find that the function f is given by

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot \cdots \cdot (3k-2)}{(3k)!}\,z^{3k}.$$

You should check that the function f is an entire function.

Here is a second approach that works well in some cases. (The previous method will always work, but it can become tedious.) We start with the differential equation and take derivatives:

$$\begin{aligned}
f''(z) &= zf(z); & f''(0) &= 0; \\
f'''(z) &= zf'(z) + f(z); & f'''(0) &= f(0) = 1; \\
f^{(4)}(z) &= zf''(z) + 2f'(z); & f^{(4)}(0) &= 2f'(0) = 0; \\
f^{(5)}(z) &= zf'''(z) + 3f''(z); & f^{(5)}(0) &= 3f''(0) = 0; \\
f^{(6)}(z) &= zf^{(4)}(z) + 4f^{(3)}(z); & f^{(6)}(0) &= 4f^{(3)}(0) = 1 \cdot 4; \\
&\vdots & &\vdots \\
f^{(k)}(z) &= zf^{(k-2)}(z) + (k-2)f^{(k-3)}(z); & f^{(k)}(0) &= (k-2)f^{(k-3)}(0);
\end{aligned}$$

By identifying a pattern, we find that $f^{(3k-2)}(0) = 0$ and $f^{(3k-1)}(0) = 0$ for all $k \geq 1$. Writing out a few more terms for the $3k$ case, we have

$$f^{(9)}(0) = 7f^{(6)}(0) = 1 \cdot 4 \cdot 7, \quad f^{(12)}(0) = 10f^{(9)}(0) = 1 \cdot 4 \cdot 7 \cdot 10, \quad f^{(15)}(0) = 13f^{(12)}(0) = 1 \cdot 4 \cdot 7 \cdot 10 \cdot 13.$$

In general, we find that

$$f^{(3k)}(0) = 1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3k-2) \qquad \text{and thus} \qquad a_{3k} = \frac{f^{(3k)}(0)}{(3k)!} = \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3k-2)}{(3k)!}$$

for $k \geq 1$. This gives the same power series for f that was found earlier.

We now turn to a different type of series known as Laurent series. We begin with some examples based upon the geometric series. Note that

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z} \qquad \text{for } |z| < 1;$$

$$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{1-(1/z)} = \frac{z}{z-1} \qquad \text{for } |1/z| < 1 \quad \Leftrightarrow \quad |z| > 1.$$

We can also use the geometric series formula in the other direction:

$$\frac{1}{1-z} = \frac{-1/z}{1-(1/z)} = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \cdots \qquad \text{for } |z| > 1.$$

$$\frac{z}{z-1} = \frac{-z}{1-z} = -z\left(1 + z + z^2 + z^3 + \cdots\right) = -z - z^2 - z^3 - z^4 + \cdots \qquad \text{for } |z| < 1.$$

Putting these equations together in summation form, we find that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \quad \text{for } |z| < 1; \qquad \frac{z}{z-1} = \sum_{k=1}^{\infty} -z^k \quad \text{for } |z| > 1;$$

$$\frac{1}{1-z} = \sum_{k=1}^{\infty} -\frac{1}{z^k} \quad \text{for } |z| > 1; \qquad \frac{z}{z-1} = \sum_{k=0}^{\infty} \frac{1}{z^k} \quad \text{for } |z| > 1.$$

For Taylor series, we represent functions as infinite degree polynomials. The above examples indicate that we can also represent functions using negative integers as exponents. We are thus led to the following definition.

Definition 17: Suppose that f is analytic in an annulus $\{z : r < |z - z_0| < R\}$. A Laurent series is a series of the form $\sum\limits_{k=-\infty}^{\infty} a_k (z - z_0)^k$. We can write this series in different ways:

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k &= \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \\
&= \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \\
&= \cdots + \frac{a_{-3}}{(z - z_0)^3} + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + a_3 (z - z_0)^3 + \cdots.
\end{aligned}$$

By a theorem in the book, it turns out that $a_k = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(w)}{(w - z_0)^{k+1}}\, dw$ for each integer k, where C is a simple closed positively oriented contour that lies in the annulus and contains z_0 in its interior.

You should read the examples in Section 5.5 carefully to see how geometric series can be used to fairly easily determine Laurent series for rational functions. Finding Laurent series for other functions can be quite easy in some cases. As the next example shows, we can sometimes just use the Maclaurin series for a known function.

Example 18: Find the Laurent series for the function $f(z) = z\big(1 - \cos(1/z)\big)$ in $|z| > 0$.

Solution: Starting with the Maclaurin series for cosine, we find that

$$\begin{aligned}
\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, z^{2k} \quad &\Rightarrow \quad \cos(1/z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, z^{-2k} \\
&\Rightarrow \quad 1 - \cos(1/z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!}\, z^{-2k} \\
&\Rightarrow \quad z\big(1 - \cos(1/z)\big) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!}\, z^{-2k+1}.
\end{aligned}$$

Pay particular attention to the second step; adding 1 just cancels one term of the series. It sometimes helps to write out terms rather than work with summation notation. Doing so yields

$$\begin{aligned}
\cos z = 1 - \frac{1}{2}\, z^2 + \frac{1}{24}\, z^4 - \frac{1}{6!}\, z^6 + \cdots \quad &\Rightarrow \quad \cos(1/z) = 1 - \frac{1}{2z^2} + \frac{1}{24z^4} - \frac{1}{6!z^6} + \cdots \\
&\Rightarrow \quad 1 - \cos(1/z) = \frac{1}{2z^2} - \frac{1}{24z^4} + \frac{1}{6!z^6} - \cdots \\
&\Rightarrow \quad z\big(1 - \cos(1/z)\big) = \frac{1}{2z} - \frac{1}{24z^3} + \frac{1}{6!z^5} - \cdots.
\end{aligned}$$

You should verify that the two sums, the summation form and the long form, representing the final answer do give the same terms.