A Brief Summary of Differential Calculus

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{v\to x} \frac{f(v)-f(x)}{v-x} \quad \text{or (equivalently)} \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

for each value of x in the domain of f for which the limit exists.

The number $f'(c)$ represents the **slope** of the graph $y = f(x)$ at the point $(c, f(c))$. It also represents the **rate of change** of y with respect to x when x is near c.

An equation for the **tangent line** to the curve $y = f(x)$ when $x = c$ is $y - f(c) = f'(c)(x - c)$.

Using the definition of the derivative, it is possible to establish the following **derivative formulas**.

$$\frac{d}{dx}\,x^r = rx^{r-1},\ r \neq 0 \qquad \frac{d}{dx}\,\sin x = \cos x \qquad \frac{d}{dx}\,\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\,\ln|x| = \frac{1}{x} \qquad \frac{d}{dx}\,\cos x = -\sin x \qquad \frac{d}{dx}\,\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\,e^x = e^x \qquad \frac{d}{dx}\,\tan x = \sec^2 x \qquad \frac{d}{dx}\,\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\,\log_a|x| = \frac{1}{(\ln a)x},\ a > 0 \qquad \frac{d}{dx}\,\cot x = -\csc^2 x \qquad \frac{d}{dx}\,\text{arccot}\, x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\,a^x = (\ln a)\,a^x,\ a > 0 \qquad \frac{d}{dx}\,\sec x = \sec x \tan x \qquad \frac{d}{dx}\,\text{arcsec}\, x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\,\csc x = -\csc x \cot x \qquad \frac{d}{dx}\,\text{arccsc}\, x = -\frac{1}{|x|\sqrt{x^2-1}}$$

product rule: $\dfrac{d}{dx}\big(F(x)G(x)\big) = F(x)G'(x) + G(x)F'(x)$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{F(x)}{G(x)}\right) = \dfrac{G(x)F'(x) - F(x)G'(x)}{(G(x))^2}$

chain rule: $\dfrac{d}{dx}\,F\big(G(x)\big) = F'\big(G(x)\big)\,G'(x)$

Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

This theorem states that there is a point c for which the instantaneous rate of change of f at c ($f'(c)$) equals the average (mean) rate of change of f on $[a, b]$ ($(f(b) - f(a))/(b - a)$). You should be familiar with a graphical interpretation of this theorem; it involves two parallel lines.

The Mean Value Theorem can be used to prove the following three facts.

1. If f' is positive (negative) on an interval I, then f is **increasing** (**decreasing**) on I. This fact makes it possible to use f' to determine the values of x for which f has a **relative maximum value** or a **relative minimum value**. The first step is to find the **critical points** of f: points x in the domain of f for which either $f'(x) = 0$ or $f'(x)$ does not exist. Then the **First Derivative Test** (or perhaps the **Second Derivative Test**) can be used to determine the nature of the critical point.

2. If f'' is positive (negative) on an interval I, then f is **concave up** (**concave down**) on I. An **inflection point** occurs where the graph changes concavity. Possible inflection points occur when $f''(x) = 0$, but it is necessary to check that the concavity actually changes at such points.

3. If $f' = g'$ on an interval I, then there is a constant C such that $g(x) = f(x) + C$ for all x in I.

A function f is **continuous** at a number c if $\lim_{x \to c} f(x) = f(c)$. This fact guarantees that the graph of f does not have a break at c. An important theorem states: If f is differentiable at c, then f is continuous at c. However, the converse is false; the function $f(x) = |x|$ is continuous at 0 but not differentiable at 0.

Intermediate Value Theorem: If f is continuous on a closed interval $[a, b]$ and v is any number between $f(a)$ and $f(b)$, then there is a number c in (a, b) such that $f(c) = v$.

Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then there exist numbers c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all x in $[a, b]$. (The number $f(c)$ is the minimum value of f on $[a, b]$ and the number $f(d)$ is the maximum value of f on $[a, b]$.)

Formal definition of limit: Let f be defined on some open interval containing the point c, except possibly at c. Then $\lim_{x \to c} f(x) = L$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x that satisfy $0 < |x - c| < \delta$.

A function f has a **vertical asymptote** $x = c$ if either $\lim_{x \to c^-} |f(x)| = \infty$ or $\lim_{x \to c^+} |f(x)| = \infty$.
A function f has a **horizontal asymptote** $y = d$ if either $\lim_{x \to \infty} f(x) = d$ or $\lim_{x \to -\infty} f(x) = d$.

Various algebraic techniques (factoring, expanding, finding a common denominator, multiplying by the conjugate) can be used to evaluate limits. The following rule is sometimes useful for computing limits of the form $0/0$ or ∞/∞; these are known as **indeterminate forms**. The suitable conditions mentioned in the hypotheses involve continuity and differentiability conditions that will always be met by the functions we encounter. (For the record, other indeterminate forms include $0 \cdot \infty$, $\infty - \infty$, 1^∞, and ∞^0.)

L'Hôpital's Rule: Under suitable conditions on the functions f and g, if either $\lim_{x \to *} f(x) = 0 = \lim_{x \to *} g(x)$ or $\lim_{x \to *} f(x) = \infty = \lim_{x \to *} g(x)$, then $\lim_{x \to *} \frac{f(x)}{g(x)} = \lim_{x \to *} \frac{f'(x)}{g'(x)}$, assuming that the latter limit exists. (The limits here can be of any type; $x \to c$, $x \to c^+$, $x \to c^-$, $x \to \infty$, $x \to -\infty$.)