Chapter 2

The Fifth Postulate

"One of Euclid's postulates—his postulate 5—had the fortune to be an epoch-making statement—perhaps the most famous single utterance in the history of science." — Cassius J. Keyser[1]

10. Introduction.

Even a cursory examination of Book I of Euclid's *Elements* will reveal that it comprises three distinct parts, although Euclid did not formally separate them. There is a definite change in the character of the propositions between Proposition 26 and Proposition 27. The first twenty-six propositions deal almost entirely with the elementary theory of triangles. Beginning with Proposition 27, the middle section introduces the important theory of parallels and leads adroitly through Propositions 33 and 34 to the third part. This last section is concerned with the relations of the areas of parallelograms, triangles, and squares and culminates in the famous I.47 and its converse. In connection with our study of the common notions and postulates we have already had occasion to examine a number of the propositions of the first of the three sections. It is a fact to be noted that the Fifth Postulate was not used by Euclid in the proof of any of these propositions. They would still be valid if the Fifth Postulate were deleted or replaced by another one compatible with the remaining postulates and common notions.

Turning our attention to the second division, consisting of Propositions 27–34, we shall find it profitable to state the first three and recall their proofs.

Proposition I.27: *If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.*

Let ST (Fig. 3) be a transversal cutting lines AB and CD in such a way that angles BST and CTS are equal [labeled α in the figure]. Assume that AB and CD meet in a point P in the direction of B and D. Then, in triangle SPT, the exterior angle CTS is equal to the interior and opposite angle TSP. But this is impossible. It follows that AB and CD cannot meet in the direction of B and D. By similar argument, it can be shown that they cannot meet in the direction of A and C. Hence they are parallel.

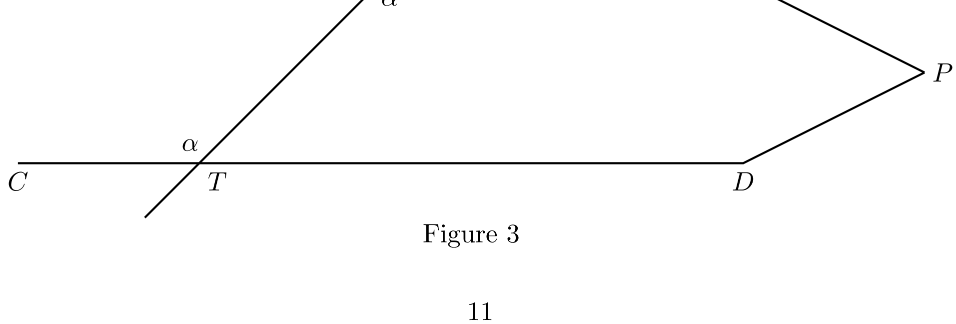

Figure 3

Proposition I.28: *If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.*

The proof, which follows easily from I.27, is left to the reader.

When we come to Proposition 29, the converse of Propositions 27 and 28, we reach a critical point in the development of Euclidean Geometry. Here, for the first time, Euclid makes use of the prolix Fifth Postulate or, as it is frequently called, the *Parallel Postulate.*

Proposition I.29: *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.*

Let AB and CD (Fig. 4) be parallel lines cut in points S and T, respectively, by the transversal ST.

Assume that angle BST is greater than angle CTS. It follows easily that the sum of angles BST and STD is greater than two right angles and consequently the sum of angles AST and CTS is less than two right angles. Then, by Postulate 5, AB and CD must meet.

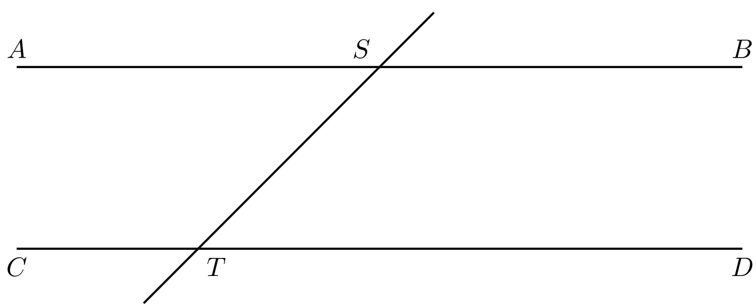

Figure 4

We conclude that angle BST cannot be greater than angle CTS. In a similar way it can be shown that angle CTS cannot be greater than angle BST. The two angles must be equal and the first part of the proposition is proved. The remaining parts are then easily verified.

There is evidence[2] that the postulates, particularly the Fifth, were formulated by Euclid himself. At any rate, the Fifth Postulate, as such, became the target for an immediate attack upon the *Elements*, an attack which lasted for two thousand years. This does not seem strange when one considers, among other things, its lack of terseness when compared with the other postulates. Technically the converse of I.17, it looks more like a proposition than a postulate and does not seem to possess to any extent that characteristic of being "self-evident." Furthermore, its tardy utilization, after so much had been proved without it, was enough to arouse suspicion with regard to its character. As a consequence, innumerable attempts were made to prove the Postulate or eliminate it by altering the definition of parallels. Of these attempts and their failures we shall have much to recount later, for they have an all-important bearing upon our subject. For the present we wish to examine some of the substitutes for the Fifth Postulate.

11. Substitutes for the Fifth Postulate.

When, in the preceding chapter, attention was directed to the importance of the Fifth Postulate in elementary geometry and in what is to follow here, the reader may have been disturbed by an inability to recall any previous encounter with the Postulate. Such a situation is due to the fact that most writers of textbooks on geometry use some substitute postulate, essentially equivalent to the Fifth, but simpler in statement. There are many such substitutes. Heath[3] quotes nine of them. The one most commonly used is generally attributed to the geometer, Playfair, although it was stated as early as the fifth century by Proclus.

12. Playfair's Axiom.

[1] *Through a given point can be drawn only one parallel to a given line.*[4]

If Playfair's Axiom is substituted for the Fifth Postulate, the latter can then be deduced as follows:

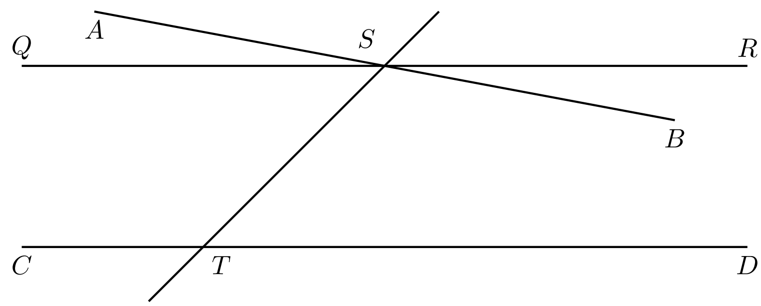

Figure 5

Given lines AB and CD (Fig. 5) cut by the transversal ST in such a way that the sum of angles BST and DTS is less than two right angles. Construct through S the line QSR, making the sum of angles RST and DTS equal to two right angles. This line is parallel to CD by I.28. Since lines QSR and ASB are different lines and, by Playfair's Axiom, only one line can be drawn through S parallel to CD, we conclude that AB meets CD. These lines meet in the direction of B and D, for, if they met in the opposite direction, a triangle would be formed with the sum of two angles greater than two right angles, contrary to I.17.

Those writers of modern textbooks on geometry who prefer Playfair's Axiom to the Fifth Postulate do so because of its brevity and apparent simplicity. But it may be contended that it is neither as simple nor as satisfactory as the Postulate. C. L. Dodgson[5] points out that there is needed in geometry a practical test by which it can be proved on occasion that two lines will meet if produced. The Fifth Postulate serves this purpose and in doing so makes use of a simple geometrical picture—two finite lines cut by a transversal and having a known angular relation to that transversal. On the other hand, Playfair's Axiom makes use of the idea of parallel lines, lines which do not meet, and about the relationship of which, within the visible portion of the plane, nothing is known. Furthermore, he shows that Playfair's Axiom asserts more than the Fifth Postulate, that "all the additional assertion is superfluous and a needless strain on the faith of the learner."

Exercises

1. Deduce Playfair's Axiom from the Fifth Postulate.
2. Prove that each of the following statements is equivalent to Playfair's Axiom:
   (a) If a straight line intersects one of two parallel lines, it will intersect the other also.
   (b) Straight lines which are parallel to the same straight line are parallel to one another.

13. The Angle-Sum of a Triangle.

A second alternative for the Fifth Postulate is the familiar theorem:

[2] *The sum of the three angles of a triangle is always equal to two right angles.*[6]

That this is a consequence of Playfair's Axiom, and hence of the Fifth Postulate, is well known. In order to deduce Playfair's Axiom from this assumption, we shall need two lemmas which are consequences of the assumption.

Lemma 1: An exterior angle of a triangle is equal to the sum of the two opposite and interior angles.

Proof: The proof is left to the reader.

Lemma 2: Through a given point P, there can always be drawn a line making with a given line ℓ an angle less than any given angle α, however small.

Proof: From P (Fig. 6) draw PA_1 perpendicular to ℓ. Measure A_1A_2 equal to PA_1 in either direction on ℓ and draw PA_2. Designate by θ_1 the equal angles A_1PA_2 and A_1A_2P. Then[7] $\theta_1 = \pi/2^2$.

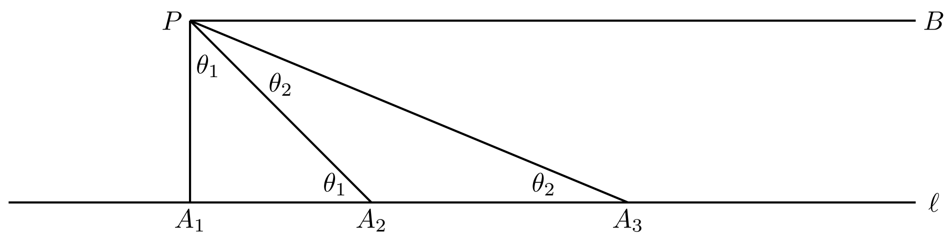

Figure 6

Next measure A_2A_3 equal to PA_2 and draw PA_3. Designate the equal angles A_3PA_2 and A_2A_3P by θ_2. Then $2\theta_2 = \theta_1$ and thus $\theta_2 = \pi/2^3$. Repeated construction leads to a triangle PA_nA_{n+1} for which its corresponding angle θ_n has measure $\pi/2^{n+1}$ and this result holds for each positive integer n. By the Postulate of Archimedes, there exists a positive integer k such that $k\alpha > \pi$. Then, if a positive integer $n > 1$ is chosen such that $2^n > k$, it follows that $\alpha > \theta_n$ and the lemma is proved.

We are now prepared to prove that, if the sum of the three angles of a triangle is always equal to two right angles, through any point can be drawn only one parallel to a given line. Let P (Fig. 6) be the given point and ℓ the given line. Draw PA_1 perpendicular to ℓ and at P draw PB perpendicular to PA_1. By I.28, PB is parallel to ℓ. Consider any line through P and intersecting ℓ, such as PA_3. Then using the fact that the sum of the angles of a triangle is π, we find that

$$\angle BPA_3 = \frac{\pi}{2} - \big(\theta_1 + \theta_2\big) = \theta_2 = \angle PA_3A_1.$$

Then PB is the only line through P which does not cut ℓ, for, no matter how small an angle a line through P makes with PB, there are, by Lemma 2, always other lines through P making smaller angles with PB and cutting ℓ, so that the first line must also cut ℓ by the Axiom of Pasch.

14. The Existence of Similar Figures.

The following statement is also equivalent to the Fifth Postulate and may be substituted for it, leading to the same consequences.

[3] *There exists a pair of similar triangles, i.e., triangles which are not congruent, but have the three angles of one equal, respectively, to the three angles of the other.*

To show that this is equivalent to the Fifth Postulate, we need only show how to deduce the latter from it, since every student of Euclid knows that the use of the Postulate leads to a geometry in which similar figures exist.

Given two triangles ABC and $A'B'C'$ (Fig. 7) with angles A, B, and C equal, respectively, to angles A', B', and C'. Let AB be greater than $A'B'$. On AB construct AD equal to $A'B'$ and on AC construct AE equal to $A'C'$. Draw DE.

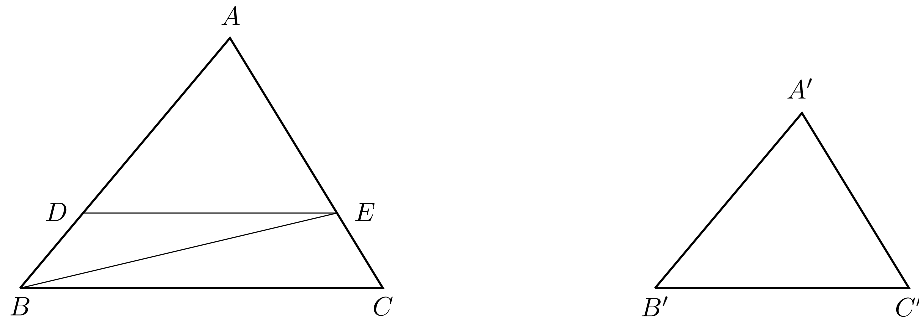

Figure 7

Then triangles ADE and $A'B'C'$ are congruent. The reader can easily show that AE is less than AC, for the assumption that AE is greater than or equal to AC leads to a contradiction. It will not be difficult now to prove that the quadrilateral $BCED$ has the sum of its four angles equal to four right angles.

Very shortly we shall prove,[8] without the use of the Fifth Postulate or its equivalent, that (a) the sum of the angles of a triangle can never be greater than two right angles, provided the straight line is assumed to be infinite, and (b) if one triangle has the sum of its angles equal to two right angles, then the sum of the angles of every triangle is equal to two right angles. By the use of these facts, our proof is easily completed.

By drawing BE, two triangles, BDE and BCE, are formed. The angle-sum for neither is greater than two right angles; if the angle-sum for either were less than two right angles, that for the other would have to be greater. We conclude that the sum of the angles for each triangle is equal to two right angles and that the same is then true for every triangle.

15. Equidistant Straight Lines.

Another noteworthy substitute is the following:

[4] *There exists a pair of straight lines everywhere equally distant from one another.*

Once the Fifth Postulate is adopted, this statement follows, for then all parallels have this property of being everywhere equally distant [see I.34]. If the above statement is postulated, we can easily deduce the Fifth Postulate by first proving that there exists a triangle with the sum of its angles equal to two right angles.

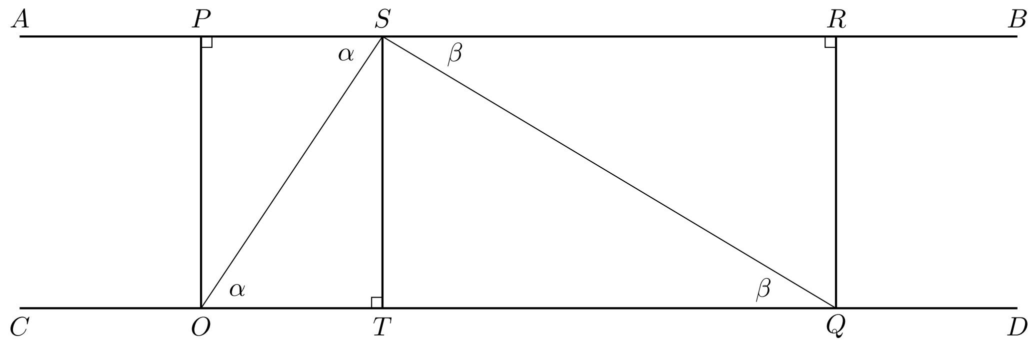

Figure 8

Let AB and CD (Fig. 8) be the two lines everywhere equally distant. From any two points O and Q on CD draw OP and QR perpendicular to AB, and from any point S on AB draw ST perpendicular to CD. By hypothesis OP, QR and ST are equal. Since right triangles OPS and OTS are congruent, $\angle PSO = \angle TOS$ [labeled α]. Similarly $\angle RSQ = \angle TQS$ [labeled β]. It follows that the sum of the angles of triangle OSQ is equal to two right angles.

16. Other Substitutes.

We conclude by stating without comment three other substitutes. The reader can show, in the light of later developments, that these are equivalent to the Fifth Postulate.

[5] *Given any three points not lying in a straight line, there exists a circle passing through them.*

[6] *If three of the angles of a quadrilateral are right angles, then the fourth angle is also a right angle.*

[7] *Through any point within an angle less than two-thirds*[9] *of a right angle there can always be drawn a straight line which meets both sides of the angle.*

These seven specimens of substitute for the Fifth Postulate are of interest as such. But they serve also to bring out the importance of the Fifth Postulate in Euclidean Geometry. Its consequences include the most familiar and most highly treasured propositions of that geometry. Without it or its equivalent there would be, for example, no Pythagorean Theorem, the whole rich theory of similar figures would disappear, and the treatment of area would have to be recast entirely. When, later on, we abandon the Postulate and replace it in turn by others which contradict it, we shall expect to find the resulting geometries strange indeed.

17. Attempts to Prove the Fifth Postulate.

We have already noted the reasons for the skepticism with which geometers, from the very beginning, viewed the Fifth Postulate as such. But the numerous and varied attempts, made throughout many centuries, to deduce it as a consequence of the other Euclidean postulates and common notions, stated or implied, all ended unsuccessfully. Before we are done we shall show why failure was inevitable. Today we know that the Postulate cannot be so derived. But these attempts, futile in so far as the main objective was concerned, are not to be ignored. Naturally it was through them that at last the true nature and significance of the Postulate were revealed. For this reason we shall find it profitable to give brief accounts of a few of the countless efforts to prove the Fifth Postulate.

18. Ptolemy.

A large part of our information about the history of Greek geometry has come to us through the writings of the philosopher, mathematician, and historian, Proclus (410–485 C.E.). He tells us that Euclid lived during the sovereignty of the first Ptolemy and that the latter himself wrote a book on the Fifth Postulate, including a proof. This must have been one of the earliest attempts to prove the Postulate. Proclus does not reproduce the proof, but from his comments we know that Ptolemy made use of the following argument in attempting to prove I.29, without using the Postulate.

Consider two parallel lines and a transversal. The two extensions of the lines on one side of the transversal are no more parallel than their two extensions on the other side of it. Then, if the sum of the two interior angles on one side is greater than two right angles, so also is the sum of those on the other. But this is impossible, since the sum of the four angles is equal to four right angles. In a similar way it can be argued that the sum of the interior angles on one side cannot be less than two right angles. The conclusion is obvious.

19. Proclus.

Proclus himself pointed out the fallacy in the above argument by remarking that Ptolemy really assumed that through a point only one parallel can be drawn to a given line. But this is equivalent to assuming the Fifth Postulate.

Proclus submitted a proof of his own. He attempted to prove that if a straight line cuts one of two parallel lines it will cut the other also. We already know that the Fifth Postulate follows readily from this. He proceeded thus:

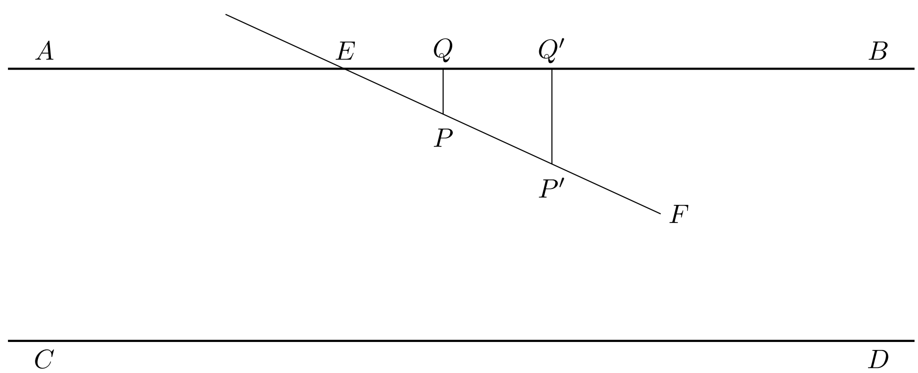

Figure 9

Given two parallel lines AB and CD (Fig. 9) with the straight line EF cutting AB at E. Assume that a point P moves along EF in the direction of F. Then the length of the perpendicular from P to AB eventually becomes greater than any length and hence greater than the distance between the parallels. Hence EF must cut CD.

The fallacy lies in the assumption that parallels are everywhere equally distant or at any rate that parallels are so related that, upon being produced indefinitely, the perpendicular from a point on one to the other remains of finite length. The former implies the Fifth Postulate, as has already been proved; the latter does also, as we shall see later.[10]

20. Nasiraddin.

For our next example we pass to the thirteenth century and consider the contributions of Nasiraddin (1201-1274), Persian astronomer and mathematician, who compiled an Arabic version of Euclid and wrote a treatise on the Euclidean postulates. He seems to have been the first to direct attention to the importance, in the study of the Fifth Postulate, of the theorem on the sum of the angles of a triangle. In his attempt to prove the Postulate one finds the germs of important ideas which were to be developed later. Nasiraddin first asserted, without proof, the following:

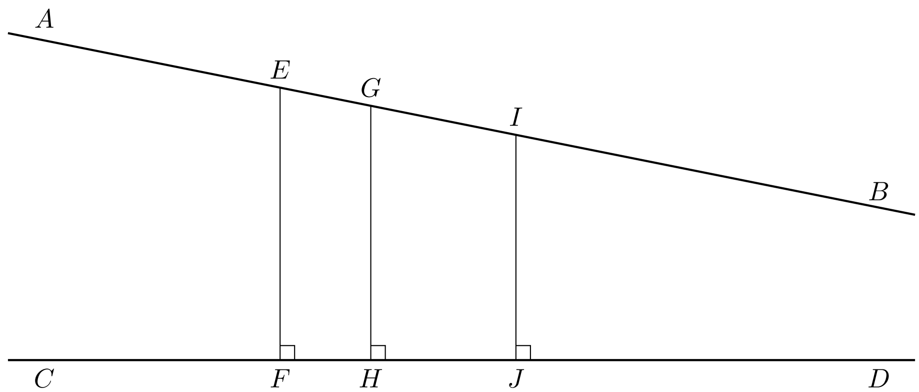

Figure 10

If two straight lines AB and CD (Fig. 10) are so related that successive perpendiculars such as EF, GH, IJ, etc., drawn to CD from points E, G, I, etc. of AB, always make unequal angles with AB, which are always acute on the side toward B, and consequently always obtuse on the side towards A, then the lines AB and CD continually diverge in the direction of A and C and, so long as they do not meet, continually converge in the direction of B and D, the perpendiculars continually growing longer in the first direction and shorter in the second. Conversely, if the perpendiculars continually become longer in the direction of A and C and shorter in the direction of B and D, the lines diverge in the first direction and converge in the other, and the perpendiculars will make with AB unequal angles, the obtuse angles all lying on the side toward A and C and the acute angles on the side towards B and D.

Next he introduced a figure destined to become famous. At the extremities of a segment AB (Fig. 11) he drew equal perpendiculars AD and BC on the same side, then joined C and D.

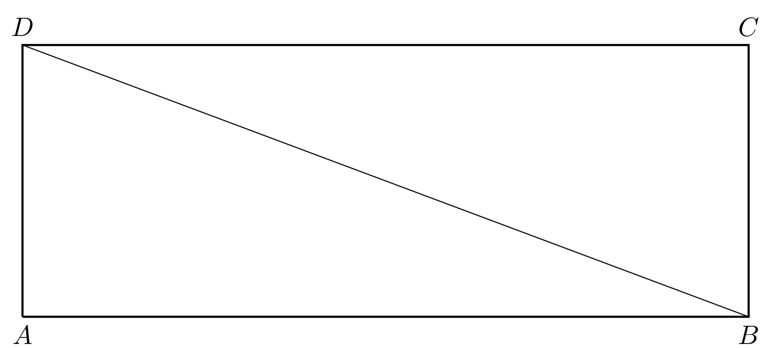

Figure 11

To prove that angles CDA and DCB are right angles, he resorted to *reductio ad absurdum*, using, without much care, the assumption stated above. Thus, if angle DCB were acute, DA would be shorter than CB, contrary to fact. Hence angle DCB is not acute. Neither is it obtuse. Of course he tacitly assumed here that, when angle DCB is acute, angle CDA must be obtuse. His argument led to the conclusion that all four angles of the quadrilateral are right angles. Then, if DB is drawn, the triangles ABD and CDB are congruent and the angle sum of each is equal to two right angles.

If everything were satisfactory so far, we know that the Fifth Postulate would follow easily. Nasiraddin himself presented an elaborate and exhaustive proof of this. But it is not difficult to pick flaws in the foregoing argument. For example, the assumptions made at the beginning are no more acceptable without proof than the Fifth Postulate itself. Again, when in Figure 11 it is assumed that angle DCB is acute, it does not follow that angle CDA is obtuse, as a matter of fact it will later be proved,[11] without use of the Fifth Postulate, that in such a figure these angles must be equal.

21. Wallis.

John Wallis (1616-1703) became interested in the work of Nasiraddin and described his demonstrations in a lecture at Oxford in 1651. In 1663 he offered a proof of his own. We describe it here because it is typical of those proofs which make use of an assumption equivalent to the Fifth Postulate.

Wallis suggested the assumption that, given a triangle, it is possible to construct another triangle similar to it and of any size. Then he argues essentially as follows:

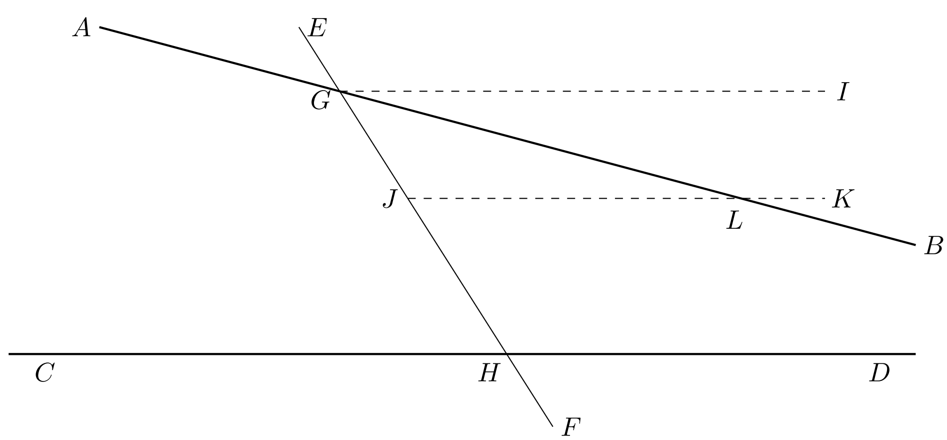

Figure 12

Given lines AB and CD (Fig. 12), cut by the transversal EF in points G and H, respectively, and with the sum of angles BGH and DHG less than two right angles. It is to be proved that AB and CD will meet if sufficiently produced.

It is easy to show that $\angle EGB > \angle GHD$. Then, if segment HG is moved along EF, with HD rigidly attached to it, until H coincides with the initial position of G, HD takes the position GI, lying entirely above GB. Hence, during its motion, HD must at some time cut GB as, for example, when it coincides with JK, cutting GB at L. Now if one constructs a triangle on base GH similar to triangle GJL — and this has been assumed to be possible — it is evident that HD must cut GB.

22. Saccheri.

In the next chapter we shall learn of the discovery of Non-Euclidean Geometry by Bolyai and Lobachewsky early in the nineteenth century. However, this discovery had all but been made by an Italian Jesuit priest almost one hundred years earlier. In 1889 there was brought to light a little book which had been published in Milan in 1733 and long since forgotten. The title of the book was *Euclides ab omni naevo vindicatus*[12] (Euclid Freed of Every Flaw), and the author was Gerolamo Saccheri (1667–1733), Professor of Mathematics at the University of Pavia.

While teaching grammar and studying philosophy at Milan, Saccheri had read Euclid's *Elements* and apparently had been particularly impressed by his use of the method of *reductio ad absurdum.* This method consists of assuming, by way of hypothesis, that a proposition to be proved is false; if an absurdity results, the conclusion is reached that the original proposition is true. Later, before going to Pavia in 1697, Saccheri taught philosophy for three years at Turin. The result of these experiences was the publication of an earlier volume, a treatise on logic. In this, his *Logica demonstrativa*, the innovation was the application of the ancient, powerful method described above to the treatment of formal logic.

It was only natural that, in casting about for material to which his favorite method might be applied, Saccheri should eventually try it out on that famous and baffling problem, the proof of the Fifth Postulate. So far as we know, this was the first time anyone had thought of denying the Postulate, of substituting for it a *contradictory* statement in order to observe the consequences.

Saccheri was well prepared to undertake the task. In his *Logica demonstrativa* he had dealt ably and at length with such topics as definitions and postulates. He was acquainted with the work of others who had attempted to prove the Postulate, and had pointed out the flaws in the proofs of Nasiraddin and Wallis. As a matter of fact, it was essentially Saccheri's proof which we used above to show that the assumption of Wallis is equivalent to the Postulate.

To prepare for the application of his method, Saccheri made use of a figure with which we are already acquainted. This is the isosceles quadrilateral with the two base angles right angles.

Assuming that, in quadrilateral $ABCD$ (Fig. 11), AD and BC were equal and that the angles at A and B were right angles, Saccheri easily proved, without using the Fifth Postulate or its consequences, that the angles at C and D were equal and that the line joining the midpoints of AB and DC was perpendicular to both lines. We do not reproduce his proofs here, because we shall have to give what is equivalent to them later on. Under the Euclidean hypothesis, the angles at C and D are known to be right angles. An assumption that they are acute or obtuse would imply the falsity of the Postulate. This was exactly what Saccheri's plan required. He considered three hypotheses, calling them the *hypothesis of the right angle*, the *hypothesis of the obtuse angle* and the *hypothesis of the acute angle.* Proceeding from each of the latter two assumptions, he expected to reach a contradiction. He stated and proved a number of general propositions of which the following are among the more important:

1. *If one of the hypotheses is true for a single quadrilateral, of the type under consideration, it is true for every such quadrilateral.*
2. *On the hypothesis of the right angle, the obtuse angle or the acute angle, the sum of the angles of a triangle is always equal to, greater than or less than two right angles.*
3. *If there exists a single triangle for which the sum of the angles is equal to, greater than or less than two right angles, then follows the truth of the hypothesis of the right angle, the obtuse angle or the acute angle.*

4. *Two straight lines lying in the same plane either have (even on the hypothesis of the acute angle) a common perpendicular or, if produced in the same direction, either meet one another once at a finite distance or else continually approach one another.*

Making Euclid's tacit assumption that the straight line is infinite, Saccheri had no trouble at all in disposing of the hypothesis of the obtuse angle. Upon this hypothesis he was able to prove the Fifth Postulate, which in turn implies that the sum of the angles of a triangle is equal to two right angles, contradicting the hypothesis. It will be seen later, however, that if he had not assumed the infinitude of the line, as he did in making use of Euclid I.18 in his argument, the contradiction could never have been reached.

But the hypothesis of the acute angle proved more difficult. The expected contradiction did not come. As a matter of fact, after a long sequence of propositions, corollaries and scholia, many of which were to become classical theorems in Non-Euclidean Geometry, Saccheri concluded lamely that the hypothesis leads to the absurdity that there exist two straight lines which, when produced to infinity, merge into one straight line and have a common perpendicular at infinity. One feels very sure that Saccheri himself was not thoroughly convinced by a demonstration involving such hazy concepts. Indeed, it is significant that he tried a second proof, though with no greater success. Had Saccheri suspected that he had reached no contradiction simply because there was none to be reached, the discovery of Non-Euclidean Geometry would have been made almost a century earlier than it was. Nevertheless, his is really a remarkable work. If the weak ending is ignored, together with a few other defects, the remainder marks Saccheri as a man who possessed geometric skill and logical penetration of high order. It was he who first had a glimpse of the three geometries, though he did not know it. He has been aptly compared with his fellow countryman, Columbus, who went forth to discover a new route to a known land, but ended by discovering a new world.

23. Lambert.

In Germany, a little later, Johann Heinrich Lambert (1728–1777) also came close to the discovery of Non-Euclidean Geometry. His investigations on the theory of parallels were stimulated by a dissertation by Georgius Simon Klügel which appeared in 1763. It appears that Klügel was the first to express some doubt about the possibility of proving the Fifth Postulate.

There is a striking resemblance between Saccheri's *Euclides Vindicatus* and Lambert's *Theorie der Parallelinien*,[13] which was written in 1766, but appeared posthumously. Lambert chose for his fundamental figure a quadrilateral with three right angles, that is, one-half the isosceles quadrilateral used by Saccheri. He proposed three hypotheses in which the fourth angle of this quadrilateral was in turn right, obtuse and acute. In deducing propositions under the second and third hypotheses, he was able to go much further than Saccheri. He actually proved that the area of a triangle is proportional to the difference between the sum of its angles and two right angles, to the excess in the case of the second hypothesis and to the deficit in the case of the third. He noted the resemblance of the geometry based on the second hypothesis to spherical geometry in which the area of a triangle is proportional to its spherical excess, and was bold enough to lean toward the conclusion that in a like manner the geometry based on the third hypothesis could be verified on a sphere with imaginary radius. He even remarked that in the third case there is an absolute unit of length.

He, like Saccheri, was able to rule out the geometry of the second hypothesis, but he made the same tacit assumptions without which no contradictions would have been reached. His final conclusions for the third geometry were indefinite and unsatisfactory. He seemed to realize that the arguments against it were largely the results of tradition and sentiment. They were, as he said, *argumenta ab amore et invidia ducta*, arguments of a kind which must be banished altogether from geometry, as from all science.

One cannot fail to note that, while geometers at this time were still attempting to prove the Postulate, nevertheless they were attacking the problem with more open minds. The change had been slow, but there is no doubt that old prejudices were beginning to disappear. The time was almost ripe for far-reaching discoveries to be made.

24. Legendre.

Finally, we must not fail to include, in our discussion of the attempts to prove the Postulate, some account of the extensive writings of Adrien Marie Legendre (1752–1833). Not that he made any valuable original contribution to the subject, for most of his results had already been obtained substantially by his predecessors. But the simple, straightforward style of his proofs brought him a large following and helped to create an interest in these ideas just at a time when geometers were on the threshold of great discoveries. Some of his proofs, on account of their elegance, are of permanent value.

His attack upon the problem was much like Saccheri's and the results which he obtained were to a large extent the same. He chose, however, to place emphasis upon the angle-sum of the triangle and proposed three hypotheses in which the sum of the angles was, in turn, equal to, greater than, and less than two right angles, hoping to be able to reject the last two. Unconsciously assuming the straight line infinite, he was able to eliminate the geometry based on the second hypothesis by proving the following theorem:

The sum of the three angles of a triangle cannot be greater than two right angles.

Assume that the sum of the angles of a triangle ABC (Fig. 13) is $180° + \epsilon$ and that angle CAB is not greater than either of the others.

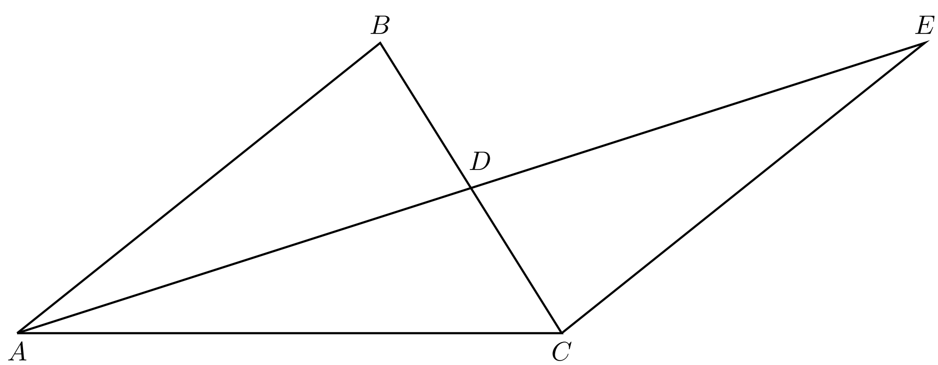

Figure 13

Join A to D, the midpoint of BC, and produce AD to E so that DE is equal to AD. Draw CE. Then triangles BDA and CDE are congruent. It follows easily that the sum of the angles of triangle AEC is equal to the sum of the angles of triangle ABC, namely to $180° + \epsilon$, and that one of the angles CAE and CEA is equal to or less than one-half angle CAB. By applying the same process to triangle AEC, one obtains a third triangle with angle-sum equal to $180° + \epsilon$ and one of its angles equal to or less than $\frac{1}{4}\angle CAB$. When this construction has been made n times, a triangle is reached which has the sum of its angles equal to $180° + \epsilon$ and one of its angles equal to or less than $2^{-n}\angle CAB$.

By the Postulate of Archimedes, we know that there is a finite multiple of ϵ, however small ϵ may be, which exceeds angle CAB, i.e., $\angle CAB < k\epsilon$. If n is chosen so large that $k < 2^n$, then $2^{-n}\angle CAB < \epsilon$, and the sum of two of the angles of the triangle last obtained is greater than two right angles. But that is impossible [see I.17].

One recognizes at once the similarity of this proof to that of Euclid I.16. Here also one sees how important for the proof is the assumption of the infinitude of the line.

But, although he made numerous attempts, Legendre could not dispose of the third hypothesis. This, as Gauss remarked, was the reef on which all the wrecks occurred. We know now that these efforts were bound to be futile. It will be of interest, however, to examine one of his attempted proofs that the sum of the angles of a triangle cannot be less than two right angles.

Assume that the sum of the three angles of triangle ABC (Fig. 14) is $180° - \epsilon$ and that angle BAC is not greater than either of the others [so the measure of this angle is less than $60°$].

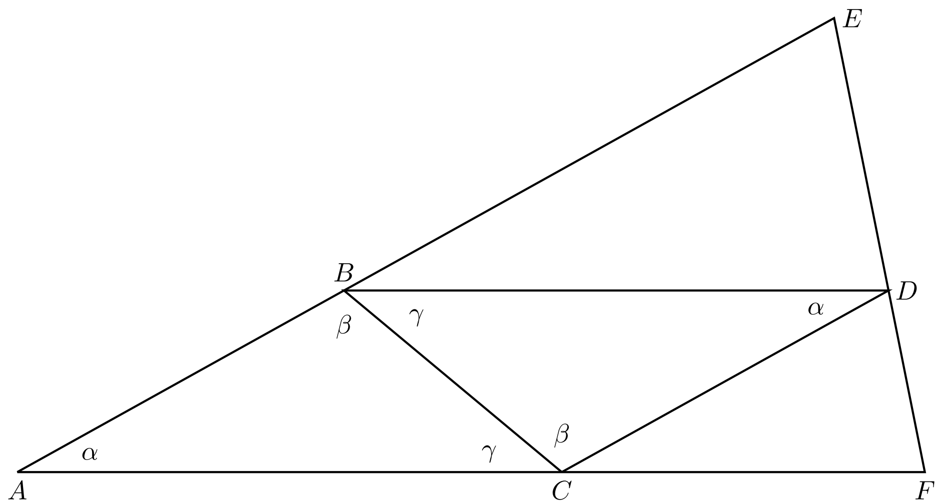

Figure 14

Construct on side BC a triangle BCD congruent to triangle ABC, with angles DBC and DCB equal, respectively, to angles BCA and CBA [denoted by γ and β]. Then draw through D any line which cuts AB and AC produced in E and F, respectively.

The sum of the angles of triangle BCD is also $180° - \epsilon$. Since, as proved above, the sum of the angles of a triangle cannot be greater than two right angles, the sum of the angles of triangle BDE and also of triangle CDF cannot be greater than $180°$. Then the sum of all of the angles of all four triangles cannot be greater than $720° - 2\epsilon$. It follows that the sum of the three angles of triangle AEF cannot be greater than $180° - 2\epsilon$.

If this construction is repeated until n such triangles have been formed in turn, the last one will have its angle sum not greater than $180° - 2^n\epsilon$. But, since a finite multiple of ϵ can be found which is greater than two right angles, n can be chosen so large that a triangle will be reached which has the sum of its angles negative, and this is absurd.

The fallacy in this proof lies in the assumption that, through any point within an angle less than two-thirds of a right angle, there can always be drawn a straight line which meets both sides of the angle. This is equivalent, as we have already remarked, to the assumption of the Fifth Postulate.

The proofs of the following sequence of important theorems are essentially those of Legendre.

If the sum of the angles of a triangle is equal to two right angles, the same is true for all triangles obtained from it by drawing lines through vertices to points on the opposite sides.

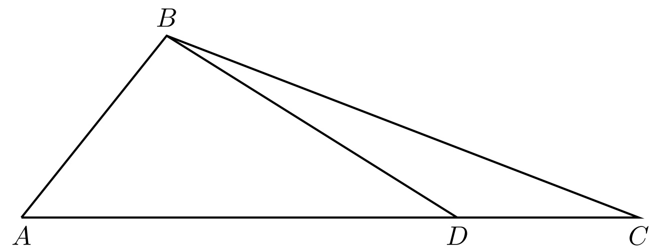

Figure 15

If the sum of the angles of triangle ABC (Fig. 15) is equal to two right angles, then the same must be true for triangle ABD, one of the two triangles into which triangle ABC is subdivided by the line joining vertex B to point D on the opposite side. For the sum of the angles of triangle ABD cannot be greater than two right angles (as proved above, with the tacit assumption of the infinitude of the straight line), and if the sum were less than two right angles, that for triangle BDC would have to be greater than two right angles [since the sum of all six angles is $360°$].

If there exists a triangle with the sum of its angles equal to two right angles, an isosceles right triangle can be constructed with the sum of its angles equal to two right angles and the legs greater in length than any given line segment.

Let the sum of the angles of triangle ABC (Fig. 16) be equal to two right angles. If ABC is not an isosceles right triangle, such a triangle, with the sum of its angles equal to two right angles, can be constructed by drawing altitude BD and then, if neither of the

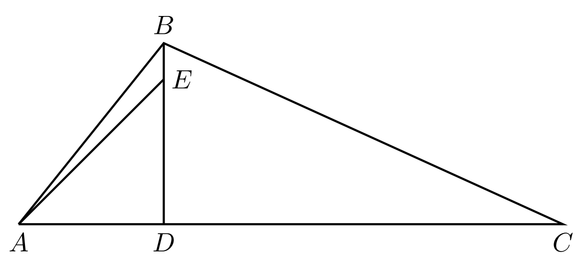

Figure 16

resulting right triangles is isosceles, measuring off on the longer leg of one of them a segment equal to the shorter. For example, if BD is greater than AD, measure DE equal to AD and draw AE. [The angle-sum is still $180°$ by the previous result.]

If two such isosceles right triangles which are congruent are adjoined in such a way that the hypotenuse of one coincides with that of the other, a quadrilateral will be formed with its angles all right angles and its sides equal. With four such congruent quadrilaterals there can be formed another of the same type with its sides twice as long as those of the one first obtained. If this construction is repeated often enough, one eventually obtains, after a finite number of operations, a quadrilateral of this kind with its sides greater than any given line segment [note the use of the Archimedean property]. A diagonal of this quadrilateral divides it into two right triangles of the kind described in the theorem.

If there exists a single triangle with the sum of its angles equal to two right angles, then the sum of the angles of every triangle will be equal to two right angles.

Given a triangle with the sum of its three angles equal to two right angles, it is to be proved that any other triangle ABC has its angle sum equal to two right angles. It may be assumed that ABC (Fig. 17) is a right triangle, since any triangle can be divided into two right triangles.

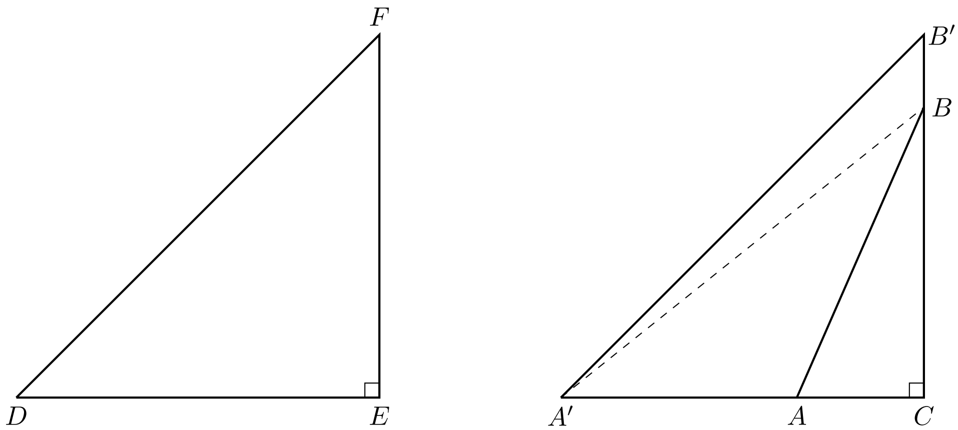

Figure 17

By the preceding theorem, there can be constructed an isosceles right triangle DEF, with the sum of its three angles equal to two right angles and its equal legs greater than the legs of triangle ABC. Produce CA and CB to A' and B', respectively, so that $CA' = CB' = ED = EF$, and join A' to B and to B' . Since triangles $A'CB'$ and DEF are congruent, the former has the sum of its angles equal to two right angles and the same is true for triangle $A'BC$ and finally for ABC.

As an immediate consequence of these results, Legendre obtained the theorem:

If there exists a single triangle with the sum of its angles less than two right angles, then the sum of the angles of every triangle will be less than two right angles.

25. Some Fallacies in Attempts to Prove the Postulate.

Of the so-called proofs of the Fifth Postulate already considered, some have depended upon the conscious or unconscious use of a substitute, equivalent to the Postulate in essence, and have thus begged the question. Others have made use of the *reductio ad absurdum* method, but in each case with results which have been nebulous and unconvincing. But there are other types of attempted proof. Some of them are very ingenious and seem quite plausible, with fallacies which are not easy to locate. We shall conclude this chapter by examining two of them.

26. The Rotation Proof.

This ostensible proof, due to Bernhard Friedrich Thibaut[14] (1775-1831) is worthy of note because it has from time to time appeared in elementary texts and has otherwise been endorsed. The substance of the proof is as follows:

In triangle ABC (Fig. 18 [black triangle]), allow side AB to rotate about A, clockwise,

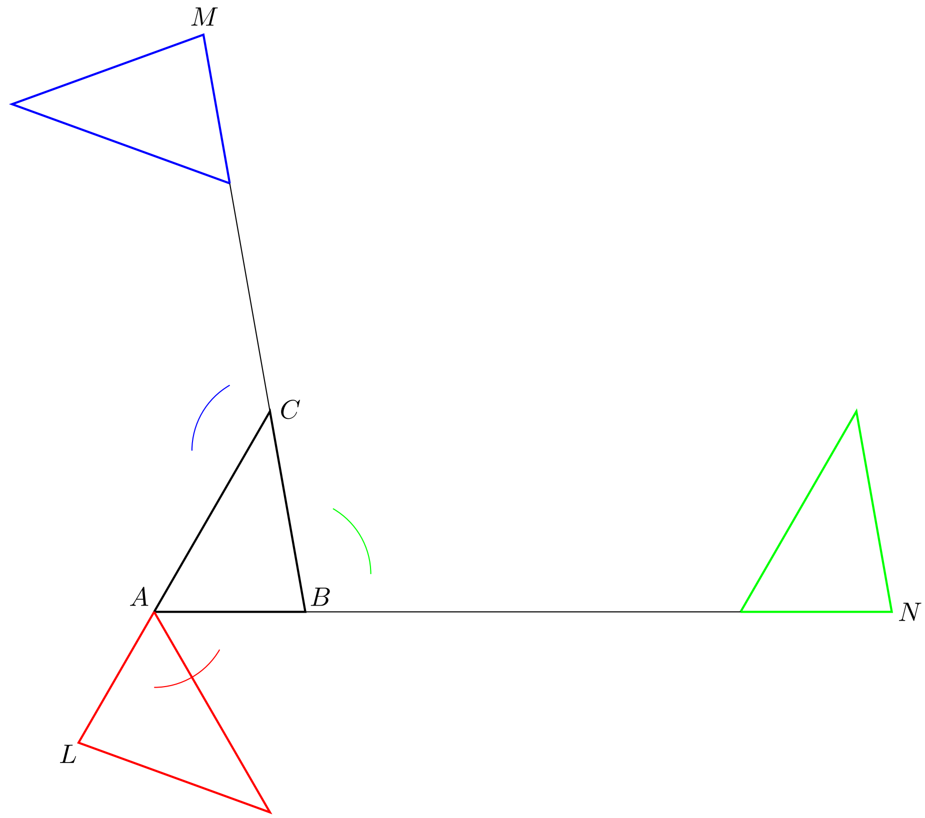

Figure 18

until it coincides with CA produced to L [red triangle]. Let CL rotate clockwise about C until it coincides with BC produced to M [blue triangle]. Finally, when BM has been rotated clockwise about B, until it coincides with AB produced to N [green triangle], it appears that AB has undergone a complete rotation through four right angles. But the three angles of rotation are the three exterior angles of the triangle, and since their sum is equal to four right angles, the sum of the interior angles must be equal to two right angles.

This proof is typical of those which depend upon the idea of direction. The circumspect reader will observe that the rotations take place about different points on the rotating line, so that not only rotation, but translation, is involved. In fact, one sees that the segment AB, after the rotations described, has finally been translated along AB through a distance equal to the perimeter of the triangle. Thus it is assumed in the proof that the translations and rotations are independent, and that the translations may be ignored. But this is only true in Euclidean Geometry and its assumption amounts to taking for granted the Fifth Postulate. The very same argument can be used for a spherical triangle, with the same conclusion, although the sum of the angles of any such triangle is always greater than two right angles.

The proof does not become any more satisfactory if one attempts to make the rotations about a single point, say A. For if PQ is drawn through A by making angle PAL equal to angle MCA [Figure 18′], one must not conclude that angle PAB will equal angle CBN. This would, as Gauss[15] pointed out, be equivalent to the assumption that if two straight lines intersect two given lines and make equal corresponding angles with one of them, then they must make equal corresponding angles with the other also.

But this will be recognized as essentially the proposition to be proved. For if two straight lines

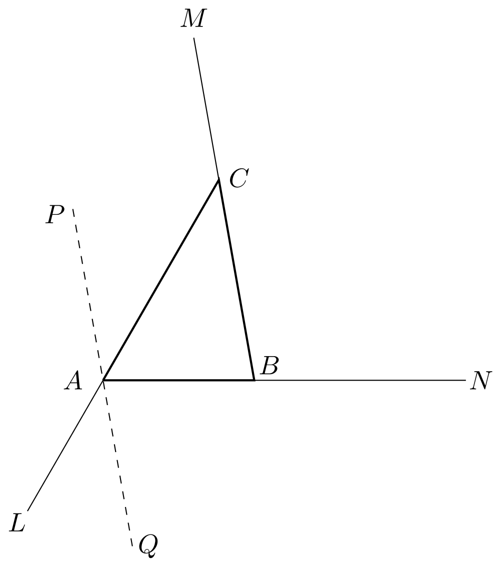

Figure 18′

make equal corresponding angles with a third, they are parallel by Euclid I.28. To conclude that they make equal angles with any other line which intersects them amounts to the assumption of I.29.

27. Comparison of Infinite Areas.

Another proof, which has from time to time captured the favor of the unwary, is due to the Swiss mathematician, Louis Bertrand[16] (1731-1812). He attempted to prove the Fifth Postulate directly, using in essence the following argument:

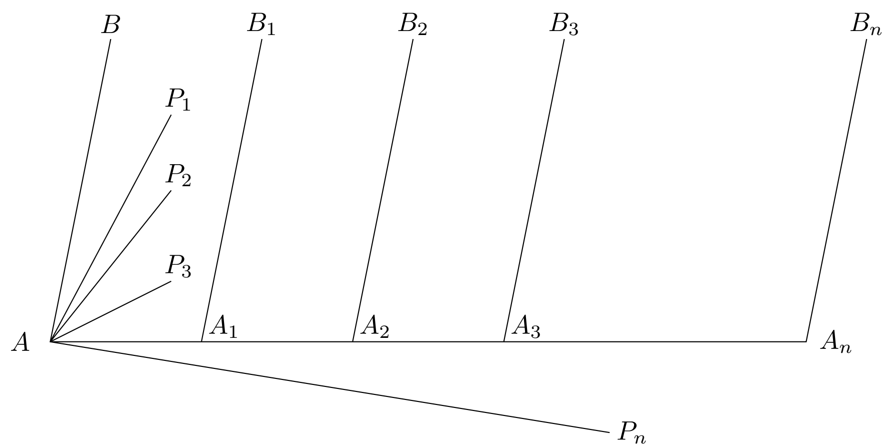

Figure 19

Given two lines AP_1 and A_1B_1 (Fig. 19) cut by the transversal AA_1 in such a way that the sum of angles P_1AA_1 and AA_1B_1 is less than two right angles, it is to be proved that AP_1 and A_1B_1 meet if sufficiently produced.

Construct AB so that angle BAA_1 is equal to angle $B_1A_1A_2$, where A_2 is a point on AA_1 produced through A_1. Then AP_1 will lie within angle BAA_1, since angle P_1AA_1 is less than angle $B_1A_1A_2$. Construct $AP_2, AP_3, \ldots, AP_n$ so that angles $P_1AP_2, P_2AP_3, \ldots, P_{n-1}AP_n$ are all equal to angle BAP_1. Since an integral multiple of angle BAP_1 can be found which exceeds angle BAA_1, n can be chosen so large that AP_n will fall below AA_1 and angle BAP_n be greater than angle BAA_1. Since the infinite sectors $BAP_1, P_1AP_2, \ldots, P_{n-1}AP_n$ can be superposed, they have equal areas and each has an area equal to that of the infinite sector BAP_n divided by n.

Next, on AA_1 produced through A_1, measure $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ all equal to AA_1, and construct $A_2B_2, A_3B_3, \ldots, A_nB_n$ so that they make with AA_n the same angle which A_1B_1 makes with that line. Then the infinite strips $BAA_1B_1, B_1A_1A_2B_2, \ldots, B_{n-1}A_{n-1}A_nB_n$ can be superposed and thus have equal areas, each equal to the area of the infinite strip BAA_nB_n divided by n. Since the infinite sector BAP_n includes the infinite strip BAA_nB_n, it follows that the area of the sector BAP_1 is greater than that of the strip BAA_1B_1, and therefore AP_1 must intersect A_1B_1 if produced sufficiently far.

The fallacy lies in treating infinite magnitudes as though they were finite. In the first place, the idea of congruence as used above for infinite areas has been slurred over and not even defined. Again, one should note that reasoning which is sound for finite areas need not hold for those which are infinite. In order to emphasize the weakness of the proof, one may compare, using the same viewpoint, the areas of the infinite sectors BAA_n and $B_1A_1A_n$. Since these sectors can be superposed, one might as a consequence conclude that they have equal areas. On the other hand, the former appears to be larger than the latter and to differ from it by the area of the infinite strip BAA_1B_1. As a matter of fact, any comparison of infinite magnitudes must ultimately be made to depend upon the process of finding the limit of a fraction, both the numerator and the denominator of which become infinite.

Footnotes

1. This quotation, as well as the one at the beginning of Chapter VIII, is taken from C. J. Keyser's book, *Mathematical Philosophy*, by permission of E. P. Dutton and Company, the publishers.
2. See Heath, *loc. cit.*, Vol. I, p. 202.
3. *Loc. cit.*, Vol. I, p. 220.
4. That *one* parallel can always be drawn follows from I.27 and I.28.
5. *Euclid and His Modern Rivals*, pp. 40–47, 2nd edition (London, 1885).
6. As a matter of fact, the assumption does not have to be so broad. It is sufficient to assume that there exists *one* triangle for which the angle-sum is two right angles. See Section 24.
7. The letter π is used here to designate two right angles.
8. See Section 24.
9. See Section 24.
10. See Section 47.
11. See Section 42.
12. This book was divided into two parts, the first and more important of which is now available in English translation: Halsted, *Girolamo Saccheri's Euclides Vindicatus*, (Chicago, 1920), or see David Eugene Smith, *A Source Book in Mathematics*, p. 351 (New York, 1929).
13. This tract, as well as Saccheri's treatise, is reproduced in Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss* (Leipzig, 1895).
14. *Grundriss der reinen Mathematik*, 2nd edition (Göttingen, 1809).
15. See his correspondence with Schumacher, Engel and Stäckel, *loc. cit.*, pp. 227–230.
16. Développement nouveau de la partie élémentaire des Mathématiques, Vol. II, p. 19 (Geneva, 1778).