Calculus I
Answers to Sample Exam 2

1. Compute $\dfrac{d}{dx}\sin(x) + \cos(2x)$.

   Answer. $\cos x - 2\sin(2x)$

2. Compute $\dfrac{d}{dx}3^x$.

   Answer. $\dfrac{d}{dx}3^x = \dfrac{d}{dx}(e^{\ln(3)})^x = \dfrac{d}{dx}e^{x\ln(3)} = e^{x\ln(3)}\ln(3) = 3^x\ln(3)$

3. Compute $\dfrac{d}{dx}x\ln x$.

   Answer. $\dfrac{d}{dx}x\ln x = x\dfrac{1}{x} + 1\cdot\ln x = 1 + \ln x$

4. Compute $\dfrac{d}{dx}\arctan(e^x)$.

   Answer. $\dfrac{d}{dx}\arctan(e^x) = \dfrac{1}{1+(e^x)^2}e^x = \dfrac{e^x}{1+e^{2x}}$

5. Compute $\dfrac{d}{dx}\tan(\cos x)$.

   Answer. $\dfrac{d}{dx}\tan(\cos x) = \sec^2(\cos x)(-\sin x) = -\sec^2(\cos x)\sin x$

6. Compute $\dfrac{d}{dx}(\sin x)^{2x}$.

   Answer.
$$\begin{aligned}\frac{d}{dx}(\sin x)^{2x} &= \frac{d}{dx}(e^{\ln(\sin x)})^{2x} = \frac{d}{dx}e^{2x\ln(\sin x)}\\ &= e^{2x\ln(\sin x)}(2x\frac{\cos x}{\sin x} + 2\ln(\sin x))\\ &= (\sin x)^{2x}(2x\frac{\cos x}{\sin x} + 2\ln(\sin x))\end{aligned}$$

7. Compute the derivative y' if $\cos(x)\sin(y) = (xy)^2$

Answer.

$$\begin{aligned} \cos(x)\sin(y)y' - \sin(x)\sin(y) &= 2xy(xy' + y) \\ \cos(x)\cos(y)y' - 2x^2yy' &= 2xy^2 + \sin(x)\sin(y) \\ (\cos(x)\cos(y) - 2x^2y)y' &= 2xy^2 + \sin(x)\sin(y) \\ y' &= \frac{2xy^2 + \sin(x)\sin(y)}{\cos(x)\cos(y) - 2x^2y} \end{aligned}$$

8. Find $\lim_{x\to 0} \frac{\cos(x) - 1}{\tan x}$.

Answer. Using L'Hôpital's rule: $\lim_{x\to 0} \frac{\cos(x) - 1}{\tan x} = \lim_{x\to 0} \frac{-\sin(x)}{\sec^2 x} = \frac{0}{1} = 0$

9. Determine local maximum and minimum points and inflection points for $f(x) = \frac{x^3}{1 - x^2}$. The derivatives of f are

$$f'(x) = \frac{x^2(3 - x^2)}{(1 - x^2)^2} \qquad f''(x) = \frac{2x(3 + x^2)}{(1 - x^2)^3}$$

Answer. $f'(x) = \frac{x^2(3 - x^2)}{(1 - x^2)^2} = 0$ when x is 0, $\sqrt{3}$, or $-\sqrt{3}$. It is undefined at $x = \pm 1$, but $f(x)$ is as well, so there is nothing to check there. $f''(0) = 0$, so we check $f'(-1/2) > 0$ and $f'(1/2) > 0$, so there is no maximum or minimum at 0. $f''(\sqrt{3}) < 0$, so there is a local maximum at $(\sqrt{3}, -3\sqrt{(3)}/2)$. $f''(-\sqrt{3}) > 0$, so there is a local minimum at $(-\sqrt{3}, 3\sqrt{(3)}/2)$.

$f''(x) = 0$ when $x = 0$. We check $f''(-1/2) < 0$ and $f''(1/2) > 0$, so there is an inflection point at $(0, 0)$.

10. Sketch the graph of $f(x) = x^4 - 4x^3 + 16x$. The derivatives of f are

$$f'(x) = 4(x+1)(x-2)^2 \qquad f''(x) = 12x(x-2)$$

Answer. $f'(x) = 0$ when x is -1 or 2. $f''(-1) > 0$, so there is a local minimum at $(-1, -11)$. $f''(2) = 0$, so we check $f'(1) > 0$ and $f'(3) > 0$, so there is no maximum or minimum at $x = 2$.

$f''(x) = 0$ when x is 0 or 2. We know $f''(-1) > 0$; we compute $f''(1) < 0$ and $f''(3) > 0$, so there are inflection points at $(0, 0)$ and $(2, 16)$.