Calculus I
Sample Exam 3

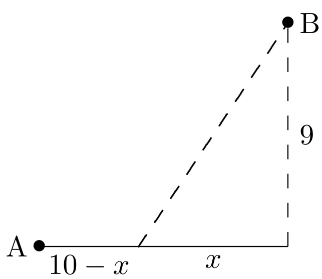
1. Find the maximum and minimum points (x and y coordinates) of $f(x) = x - \cos x$ on $[-\pi, \pi]$.

   Answer. The derivative is $f'(x) = 1 + \sin x$; the only critical value in the interval is $-\pi/2$. Computing $f(x)$ for the critical value and endpoints gives $f(-\pi) = 1 - \pi$, $f(-\pi/2) = -\pi/2$, $f(\pi) = \pi + 1$. The maximum point is $(\pi, \pi + 1)$ and the minimum point is $(-\pi, 1 - \pi)$.

2. A cylinder is to be manufactured to hold 1 liter (1000 cm^3). Find the radius and height of the cylinder that has the smallest possible surface area (top plus bottom plus side).

   Answer. The area is $A = 2\pi r^2 + 2\pi rh$. Since the volume is 1 liter or 1000 cm^3, $1000 = \pi r^2 h$ and so $h = 1000/(\pi r^2)$. Now $A(r) = 2\pi r^2 + 2\pi r(1000/(\pi r^2)) = 2\pi r^2 + 2000/r$, $r \in (0, \infty)$. Then $A'(r) = 4\pi r - 2000/r^2$ and there is one critical value, $r = (500/\pi)^{1/3} = 5(4/\pi)^{1/3}$. Since $A''(r) = 4\pi + 4000/r^3 > 0$, there is a global minimum there. The radius of the can is $r = 5(4/\pi)^{1/3}$ cm and the height is $10(4/\pi)^{1/3}$ cm.

3. One town, labeled A, is located on a highway. Another, B, is located 9 miles north of a point 10 miles east of A. The speed limit on the highway is 50 mph, and a new road is to be built from the highway to B, as indicated. The speed limit on the new road will be 25 mph. Where should the new road join the highway to minimize the (legal) travel time from A to B?



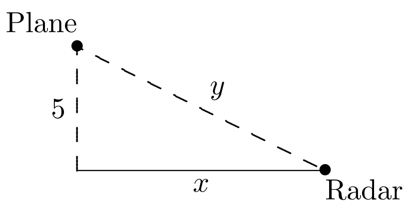
   Answer. The travel time is $T(x) = \dfrac{10 - x}{50} + \dfrac{\sqrt{x^2 + 9^2}}{25}$, $x \in [0, 10]$. The derivative is $T'(x) = -\dfrac{1}{50} + \dfrac{x}{25\sqrt{x^2 + 81}}$, so there is one critical value, $x = \sqrt{27} = 3\sqrt{3}$. The second derivative is $T''(x) = \dfrac{81}{25(x^2 + 81)^{3/2}} > 0$, so the minimum occurs at the critical value. The new road should join the highway $10 - 3\sqrt{3}$ miles from A.

4. A box with a bottom but no top is to be made from a piece of cardboard 1 foot by 2 feet, by cutting a square out of each corner. Find the shape of the box with largest volume.

   Answer. If x is the size of the cut-out squares, the volume is $V(x) = (1-2x)(2-2x)x = 4x^3 - 6x^2 + 2x$, $x \in [0, 1/2]$. The derivative is $V'(x) = 12x^2 - 12x + 2$. V' is zero when x is $\dfrac{1}{2} + \dfrac{\sqrt{3}}{6}$ or $\dfrac{1}{2} - \dfrac{\sqrt{3}}{6}$, but only the second of these is in $[0, 1/2]$. Since $V(0) = V(1/2) = 0$, the maximum value occurs at the critical value and the corresponding box is $(\sqrt{3}/3) \times (1 + \sqrt{3}/3) \times (3 - \sqrt{3})/6$.

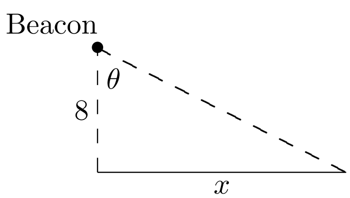
5. An airplane is flying at a constant altitude of 5 km, on a path that will take it directly over a radar dish. When the radar reports that the plane is 10 km away (measured on a straight line between plane and radar) it also reports that the distance between radar and plane is decreasing at 600 km/hr. What is the actual (that is, horizontal) speed of the plane?

   Answer. We know that $y' = -600$ when $y = 10$. Starting with $x^2 + 25 = y^2$ we get $2xx' = 2yy'$ or $x' = yy'/x$. At the time in question this means that $x' = 10(-600)/\sqrt{100-25} = -400\sqrt{3}$, so the plane is traveling at $400\sqrt{3} \approx 693$ km/hr.



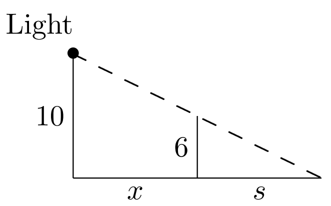
6. The light from a rotating beacon shines on a straight wall 8 feet from the beacon. The beacon makes two revolutions per minute. How fast is the spot of light on the wall moving when the spot is 10 feet from the beacon (measured in a straight line between the beacon and the spot)?

   Answer. The rotation speed of 2 revolutions per minute means that θ is changing at 4π radians per minute. Then $x/8 = \tan\theta$ so $x' = 8\sec^2(\theta) \cdot \theta'$. At the time in question this means $x' = 8(10/8)^2 4\pi = 50\pi$ feet/min.



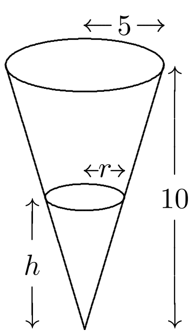
7. A man 6 ft tall walks at the rate of 3 ft/sec away from a streetlight that is 10 ft above the ground. At what rate is his shadow lengthening?

   Answer. We know that $x' = 3$ and using similar triangles we see that $s/6 = (s+x)/10$ or $10s = 6s + 6x$ or $4s = 6x$. Then $4s' = 6x'$ and $s' = (6/4)3 = 9/2$, so the shadow is lengthening at $9/2$ feet per second.



8. Water is poured into a conical container at the rate of 2 m^3/sec. The cone points directly down, and it has a height of 10 m and a base radius of 5 m. How fast is the water level rising when the water is 3 m deep (at its deepest point)?

   Answer. We know that the volume of water, V is changing at 2 m^3/sec, so $V' = 2$. The volume is $(1/3)\pi r^2 h$ and by similar triangles $r/h = 5/10$ so $r = h/2$. Thus $V = (1/3)\pi(h/2)^2 h = (1/3)\pi h^3/4$ and $V' = \pi h^2 h'/4$, so $h' = 4V'/(h^2\pi) = 4 \cdot 2/(3^2\pi) = 8/(9\pi)$ m/sec.



9. Using $x_0 = 3$ as the initial guess, compute the x_1 approximation to $\sqrt{11}$ by Newton's method.

   Answer. Using $f(x) = x^2 - 11$, $f'(x) = 2x$ and $x_1 = 3 - \dfrac{f(3)}{f'(3)} = 3 - \dfrac{-2}{6} = 10/3$.

10. The side of a square is measured at 2 kilometers, but the method of measurement is accurate to only one meter. Use differentials to estimate the largest and smallest area that the square might contain.

    Answer. The area of the square is $A = s^2$, and s is measured as $2 \cdot 10^3$ meters, corresponding to an area of $4 \cdot 10^6$ square meters. The differentials are related by $dA = 2s\,ds$. Since s might vary by ± 1 meter, $ds = \pm 1$ and so $dA = 2(2 \cdot 10^3)(\pm 1) = \pm 4 \cdot 10^3$. So the true area is approximately $4 \cdot 10^6 \pm 4 \cdot 10^3$.