Calculus II
Answers to Sample Exam 2

1. Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.

   Answer. The two curves intersect when $6x-x^2 = x^2-2x$, or $0 = 2x^2-8x$, with solutions $x = 0$ and $x = 4$. A quick sketch shows that $y = 6x - x^2$ is the larger between 0 and 4. Then the area is $\int_0^4 6x - x^2 - x^2 + 2x\,dx = \int_0^4 8x - 2x^2\,dx = 64/3$.

2. An object moves so that its acceleration at time t is given by $a(t) = -5$. Its initial position is $s(0) = 10$ and its initial velocity is $v(0) = 2$. Find $s(t)$ and $v(t)$.

   Answer. $v(t) = 2 + \int_0^t -5\,du = 2 - 5t$; $s(t) = 10 + \int_0^t 2 - 5u\,du = 10 + 2t - 5t^2/2$.

3. Find the volume generated when the area between $y = 2x^2$ and $y = 8$ from $x = 0$ to $x = 2$ is rotated around the line $y = 10$.

   Answer. $\pi \int_0^2 (10 - 2x^2)^2 - 2^2\,dx = \pi \int_0^2 4x^4 - 40x^2 + 96\,dx = \pi(\frac{4}{5}32 - \frac{40}{3}8 + 192) = \frac{1664\pi}{15}$.

4. An object moves so that its velocity is $v(t) = \sin(t)$. Find its average velocity between $t = \pi/2$ and $t = 2\pi$.

   Answer. $\frac{1}{2\pi - \pi/2} \int_{\pi/2}^{2\pi} \sin t\,dt = \frac{2}{3\pi}(-\cos t)\Big|_0^2 = -\frac{2}{3\pi}$.

5. A rope 100 meters long is hanging straight down from the top of a building. The density of the rope is 2 kg per meter. Find the amount of work required to lift the entire rope to the top of the building. Use 9.8 for the acceleration due to gravity.

   Answer. $\int_0^{100} 2 \cdot 9.8y\,dy = 98000$ N-m.

6. A spring has a natural length of 1/2 meter. It takes a force of 5 newtons to stretch the spring to 1 meter in length. Find the amount of work required to compress the spring from its natural length to 1/4 meter.

   Answer. By Hooke's Law, $k(1/2) = 5$ so $k = 10$. Then the work is $\int_0^{1/4} 10x\,dx = 5/16$, or $\int_{1/2}^{1/4} 10\left(x - \frac{1}{2}\right)\,dx = 5/16$.

7. A beam 4 meters long has density $\sigma(x) = 2 + \sin(\pi x)$, where x is the distance from the left end of the beam. Find the center of mass.

Answer. $M = \int_0^4 2 + \sin(\pi x)\, dx = 8$ and

$$\begin{aligned} M_0 &= \int_0^4 x(2+\sin(\pi x))\, dx \\ &= \int_0^4 2x\, dx + \int_0^4 x\sin(\pi x)\, dx \\ &= 16 - \frac{x\cos(\pi x)}{\pi}\Big|_0^4 + \int_0^4 \frac{\cos(\pi x)}{\pi}\, dx \\ &= 16 - \frac{4}{\pi} + 0 \\ &= \frac{4(4\pi - 1)}{\pi}, \end{aligned}$$

using integration by parts. Then $\bar{x} = \frac{M_0}{M} = \frac{4\pi - 1}{2\pi} \approx 1.84$

8. Compute $\int_1^\infty x^{-3/2}\, dx$.

Answer. $\int_1^\infty x^{-3/2}\, dx = \lim_{D\to\infty} -2x^{-1/2}\Big|_1^D = \lim_{D\to\infty} -2D^{-1/2} + 2 = 2.$

9. Set up an integral to compute the length of the curve given by $f(x) = \sin(x^2)$ for x in $[0, \pi]$. Do not evaluate the integral.

Answer. $\int_0^\pi \sqrt{1 + 4x^2\cos^2(x^2)}\, dx.$

10. Set up an integral to compute the surface area generated when $f(x) = e^x$, for x in $[0, 2]$, is rotated around the x-axis. Do not evaluate the integral.

Answer. $\int_0^2 2\pi e^x \sqrt{1 + e^{2x}}\, dx.$