Calculus II
Sample Exam 3: Chapter 11

1. Does the series described by $\sum_{n=0}^{\infty}(-1)^n \frac{n^2}{2n^2-5}$ converge?

   Answer. No. Since $\lim_{n\to\infty} \frac{n^2}{2n^2-5} = \frac{1}{2} \neq 0$, the series does not converge.

2. To what number does $\sum_{n=1}^{\infty} \frac{(-2)^n}{7^{n+1}}$ converge?

   Answer. This is a geometric series missing the first term:

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{7^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{7}\left(\frac{-2}{7}\right)^n = -\frac{1}{7} + \frac{1}{7}\sum_{n=0}^{\infty}\left(\frac{-2}{7}\right)^n = -\frac{1}{7} + \frac{1}{7}\frac{1}{1-(-2/7)} = -\frac{1}{7} + \frac{1}{9} = -\frac{2}{63}$$

3. Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ converge?

   Answer. Use the integral test, and the substitution $u = \ln x$:

$$\lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^4}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u^{-4}\,du = \lim_{t\to\infty} \frac{-1}{3u^3}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \frac{-1}{3(\ln t)^3} + \frac{1}{3(\ln 2)^3} = \frac{1}{3(\ln 2)^3},$$

   so the series converges.

4. Does $\sum_{n=0}^{\infty} \frac{n^2-2}{n^3+n+2}$ converge?

   Answer. This "looks like" the harmonic series, since $n^2/n^3 = 1/n$, so we guess that it diverges. We'd like to see that the terms are larger than terms that look like $1/n$. It is true that for large enough n:

$$\frac{n^2-2}{n^3+n+2} \geq \frac{n^2/2}{n^3+n^3+n^3} = \frac{n^2/2}{3n^3} = \frac{1}{6n},$$

   because eventually $n^2 - 2 > n^2 - n^2/2 = n^2/2$ and $n^3+n+2 < n^3+n^3+n^3 = 3n^3$. So the series diverges because $\sum 1/(6n)$ diverges.

5. Does $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converge absolutely, converge conditionally, or diverge?

Answer. It converges conditionally. It converges by the alternating series test, but it doesn't converge absolutely because $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test.

First: Since $\displaystyle\lim_{n\to\infty} \frac{1}{n \ln n} = 0$, and since $\displaystyle\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$, meaning the terms are decreasing, the series converges by the alternating series test.

Second:

$$\lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t\to\infty} \ln|u| \Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2| = \infty.$$

6. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$ converge?

Answer. Yes, by the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{1}{n+1} = 1 \cdot 0 = 0.$$

7. Find the interval of convergence and radius of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n} x^n$.

Answer. Using the ratio test:

$$\lim_{n\to\infty} \frac{3^{n+1}}{n+1} \frac{n}{3^n} |x| = 3|x|,$$

so the radius of convergence is $1/3$. When $x = 1/3$ the series is the harmonic series, so it diverges; when $x = -1/3$ the series is the alternating harmonic series, so it converges; thus the interval of convergence is $[-1/3, 1/3)$.

8. Find the interval of convergence and radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{2^n} (x-2)^n$.

Answer. Using the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^3}{2^{n+1}} \frac{2^n}{n^3} |x-2| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \frac{1}{2}|x-2| = \frac{1}{2}|x-2|,$$

so the radius of convergence is 2. When $x = 0$,

$$\sum_{n=0}^{\infty} \frac{n^3}{2^n} (0-2)^n = \sum_{n=0}^{\infty} (-1)^n n^3,$$

which diverges, by the divergence test. When $x = 4$,

$$\sum_{n=0}^{\infty} \frac{n^3}{2^n} (4-2)^n = \sum_{n=0}^{\infty} n^3,$$

which diverges, by the divergence test. Thus, the interval of convergence is $(0, 4)$.

9. Find a power series representation for x^2e^x; find the radius of convergence for your series.

   Answer. Start with the series for e^x and multiply by x^2:

$$x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}.$$

   Using the ratio test:

$$\lim_{n\to\infty} \frac{n!}{(n+1)!}|x| = \frac{1}{n+1}|x| = 0,$$

   so the radius of convergence is ∞.

10. Find a power series representation for $\cos(x^3)$; find the radius of convergence for your series.

    Answer. Starting with the series for the cosine:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n},$$

    and substituting x^3 for x:

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^3)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n}.$$

    Since the series for cosine converges for all numbers, so does the second series, so the radius of convergence is infinity.