Calculus III—Math 225

SAMPLE EXAM 1 ANSWERS

1. Let $\mathbf{a} = \langle 1, 0, -1\rangle$, $\mathbf{b} = \langle -2, 3, 5\rangle$. Find $|\mathbf{a}|$, $\mathbf{a} - \mathbf{b}$, and a unit vector in the same direction as $\mathbf{a} - \mathbf{b}$.

   Solution: $|\mathbf{a}| = \sqrt{2}$; $\mathbf{a} - \mathbf{b} = \langle 3, -3, -6\rangle$;
   unit vector is $(\mathbf{a} - \mathbf{b})/|\mathbf{a} - \mathbf{b}| = \langle 3/\sqrt{54}, -3/\sqrt{54}, -6/\sqrt{54}\rangle$.

2. Let $\mathbf{v} = \langle 5, -1, 6\rangle$, $\mathbf{w} = \langle -2, 2, -4\rangle$. Find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   Solution: $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \dfrac{-36}{\sqrt{62}\sqrt{24}}$.

3. Find a vector perpendicular to both $\langle 1, 2, 2\rangle$ and $\langle -3, 1, 5\rangle$.

   Solution: $\langle 1, 2, 2\rangle \times \langle -3, 1, 5\rangle = \langle 8, -11, 7\rangle$.

4. Find the vector projection of $\mathbf{a}$ onto $\mathbf{b}$, using $\mathbf{a} = \langle 2, 2, 2\rangle$ and $\mathbf{b} = \langle 1, -1, -1\rangle$.

   Solution: The projection is

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|} = -\frac{2}{3}\langle 1, -1, -1\rangle = \langle -\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\rangle.$$

5. Find a vector function for the line that is the intersection of the planes $x + y + z = 2$ and $3x - 2y - z = -5$.

   Solution: Adding the two equations gives $4x - y = -3$, which has solutions $x = 0, y = 3$ and $x = 1, y = 7$, so the points $(0, 3, -1)$ and $(1, 7, -6)$ are on the line. A vector parallel to the line is therefore $\langle 1, 4, -5\rangle$, and so the line is $\mathbf{r} = \langle 0, 3, -1\rangle + t\langle 1, 4, -5\rangle$.

6. Find an equation for the plane that is perpendicular to both of the planes $x+y+z = 2$ and $3x - 2y - z = -5$ and contains the point $(1, 1, 1)$.

   Solution: The normal to the plane is perpendicular to the normals of the given planes, so we may use $\langle 1, 1, 1\rangle \times \langle 3, -2, -1\rangle = \langle 1, 4, -5\rangle$. The equation of the plane is $(x - 1) + 4(y - 1) - 5(z - 1) = 0$, or $x + 4y - 5z = 0$.

7. Using $\mathbf{r}(t) = \langle t^2+2, t^2-4t, 2t\rangle$, find an equation for a plane perpendicular to $\mathbf{r}$ at $(6,-4,4)$.

Solution: We calculate $\mathbf{r}' = \langle 2t, 2t-4, 2\rangle$. Since $\mathbf{r}(2) = \langle 6,-4,4\rangle$, $\mathbf{r}'(2) = \langle 4,0,2\rangle$, and this is the normal to the plane we seek. Thus the equation for the plane is $4(x-6)+2(z-4)=0$, or $4x+2z=32$.

8. Find the curvature of $\mathbf{r}(t) = \langle t^2+2, t^2-4t, 2t\rangle$ from the previous problem as a function of t and also find the curvature at $(6,-4,4)$.

Solution: From the previous problem, $\mathbf{r}' = \langle 2t, 2t-4, 2\rangle$ and so $\mathbf{r}'' = \langle 2,2,0\rangle$. Then

$$\kappa(t) = \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle -4,4,8\rangle|}{(4t^2+(2t-4)^2+4)^{(3/2)}} = \frac{\sqrt{96}}{(4t^2+(2t-4)^2+4)^{(3/2)}},$$

$$\kappa(2) = \frac{\sqrt{96}}{(20)^{(3/2)}.}$$

9. Suppose the position of an object is given by is $\mathbf{r}(t) = \langle t^2+2, t^2-4t, 2t\rangle$. Find the scalar components of acceleration, $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$.

Solution: From the previous problem, $\kappa(t) = \dfrac{\sqrt{96}}{(4t^2+(2t-4)^2+4)^{(3/2)}}$, so $a_{\mathbf{N}} = |r'(t)|^2\kappa = \dfrac{\sqrt{96}}{(4t^2+(2t-4)^2+4)^{(1/2)}}$. Using $\mathbf{r}'$ and $\mathbf{r}''$ from the previous problem, $a_{\mathbf{T}} = \dfrac{r''\cdot r'}{|r'|} = \dfrac{4t+4t-8}{(4t^2+(2t-4)^2+4)^{(1/2)}}$.

10. Suppose an object moves so that its acceleration vector is $\langle t, t^2, \sin t\rangle$, and at $t = 0$ it is at the point $(1, 1, 1)$ with velocity $\langle -2, 1, 2\rangle$. Find the vector functions $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

Solution:

$$
\begin{aligned}
\mathbf{v}(t) &= \langle -2, 1, 2\rangle + \int_0^t \langle u, u^2, \sin u\rangle\, du \\
&= \langle -2, 1, 2\rangle + \left\langle \frac{t^2}{2}, \frac{t^3}{3}, -\cos t \right\rangle - \langle 0, 0, -1\rangle \\
&= \left\langle \frac{t^2}{2} - 2, \frac{t^3}{3} + 1, 3 - \cos t \right\rangle \\
\mathbf{r}(t) &= \langle 1, 1, 1\rangle + \int_0^t \left\langle \frac{u^2}{2} - 2, \frac{u^3}{3} + 1, 3 - \cos u \right\rangle du \\
&= \langle 1, 1, 1\rangle + \left\langle \frac{t^3}{6} - 2t, \frac{t^4}{12} + t, 3t - \sin t \right\rangle - \langle 0, 0, 0\rangle \\
&= \left\langle \frac{t^3}{6} - 2t + 1, \frac{t^4}{12} + t + 1, 3t - \sin t + 1 \right\rangle
\end{aligned}
$$