SAMPLE EXAM 2 ANSWERS

1. Let $f(x,y) = \ln(x^2+y^2)$. Compute the partial derivatives f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}.

   Solution:

   $$f_x = 2x/(x^2+y^2) \qquad f_y = 2y/(x^2+y^2) \qquad f_{xy} = f_{yx} = -4xy/(x^2+y^2)^2$$
   $$f_{xx} = (2y^2-2x^2)/(x^2+y^2)^2 \qquad f_{yy} = (2x^2-2y^2)/(x^2+y^2)^2$$

2. Describe the level curves of $f(x,y) = \dfrac{x^2}{9} + \dfrac{y^2}{16}$.

   Solution:

   All level curves are ellipses with center at the origin. The level curve $k = x^2/9 + y^2/16$ has intercepts at $x = \pm 3\sqrt{k}$ and $y = \pm 4\sqrt{k}$.

3. Find an equation for the tangent plane to $z = e^y \ln x$ at $(1,3,0)$.

   Solution:

   $f_x = e^y/x$, $f_y = e^y \ln x$. A normal to the tangent plane at $(1,3,0)$ is therefore $\langle e^3, 0, -1\rangle$, and the tangent plane is given by $z = e^3(x-1)$.

4. Use the "two-variable" version of the chain rule to compute $g'(t)$ if $g(t) = \ln(x^2+y^2)$, and $x = t^2$, $y = \cos t$. Your answer should contain all of x, y, and t.

   Solution:

   $g_x = \dfrac{2x}{x^2+y^2}$, $g_y = \dfrac{2y}{x^2+y^2}$, $x' = 2t$, and $y' = -\sin t$. So

   $$g'(t) = f_x \cdot x' + f_y \cdot y' = \frac{2x}{x^2+y^2} \cdot 2t + \frac{2y}{x^2+y^2}(-\sin t).$$

5. Suppose $z = f(x,y)$. Describe the significance of both the length and the direction of ∇f at a point (x_0, y_0).

   Solution:

   The direction of ∇f is the direction in which f increases most rapidly. The rate at which f increases most rapidly is $|\nabla f|$.

6. Find the directional derivative of $f(x, y) = x^2y^3 + xy$ at the point $(-1, 2)$ in the direction indicated by the vector $\langle 3, -2\rangle$.

Solution:

First compute $\nabla f = \langle 2xy^3 + y, 3x^2y^2 + x\rangle$, so

$$\nabla f(-1, 2) = \langle 2(-1)2^3 + 2, 3(-1)^2 2^2 - 1\rangle = \langle -14, 11\rangle.$$

A unit vector in the direction of $\langle 3, -2\rangle$ is $\langle 3, -2\rangle/\sqrt{13}$, so

$$D_{\mathbf{u}} f = \langle -14, 11\rangle \cdot \langle 3, -2\rangle/\sqrt{13} = -64/\sqrt{13}.$$

7. Find the directional derivative of $f(x, y, z) = x^2yz^3 + xy - z$ at the point $(1, 1, 1)$ in the direction indicated by the vector $\langle 1, 2, 3\rangle$.

Solution:

First compute $\nabla f = \langle 2xyz^3 + y, x^2z^3 + x, 3x^2yz^3 - 1\rangle$, so

$$\nabla f(1, 1, 1) = \langle 3, 2, 2\rangle.$$

Then $D_{\mathbf{u}} f = \langle 3, 2, 2\rangle \cdot \langle 1, 2, 3\rangle/\sqrt{14} = 13/\sqrt{14}$.

8. Find an equation for the tangent plane to the surface $12 + 1/\sqrt{2} = x^4y - x^2y^3 + y \sin z$ at the point $(2, 1, \pi/4)$.

Solution:

A normal is given by ∇f where $f(x, y, z) = x^4y - x^2y^3 + y \sin z$. Now $\nabla f = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2 + \sin z, y \cos z\rangle$, and $\nabla f(2, 1, \pi/4) = \langle 28, 4 + \sqrt{2}/2, \sqrt{2}/2\rangle$. The plane is then $28(x - 2) + (4 + \sqrt{2}/2)(y - 1) + \sqrt{2}/2(z - \pi/4) = 0$, or $28x + (4 + \sqrt{2}/2)y + (\sqrt{2}/2)z = 60 + \sqrt{2}/2 + \sqrt{2}\pi/4$.

9. The point $(1, -2, 4)$ is on the surface described by $z = x^2y^2 + y \ln x$. Find a vector in one of the two possible directions to go from this point to stay on the level curve $z = 4$.

Solution:

We compute $\nabla f = \langle 2xy^2 + y/x, 2x^2y + \ln x\rangle$ and $\nabla f(1, -2) = \langle 6, -4\rangle$. Then a vector in one of the two desired directions is $\langle 4, 6\rangle$; another correct answer is $\langle -4, -6\rangle$.

10. Find all critical points for $z = x \sin y$ and classify them as local maximum points, local minimum points, or saddle points.

Solution:

The partial derivatives are

$$f_x = \sin y \qquad f_y = x \cos y \qquad f_{xy} = f_{yx} = \cos y$$
$$f_{xx} = 0 \qquad f_{yy} = -x \sin y$$

The first partial derivatives are both zero when $x = 0$ and $y = n\pi$, for any integer n. Then

$$D(0, n\pi) = f_{xx}(0, n\pi) f_{yy}(0, n\pi) - f_{xy}(0, n\pi)^2 = 0 - \cos^2(n\pi) = -1 < 0,$$

so all are saddle points.