SAMPLE EXAM 3 SOLUTIONS

1. Compute: $\displaystyle\iint_R xe^{xy}\, dA,\ R = [0,1]\times[0,1]$.

   Solution:

$$\int_0^1 \int_0^1 xe^{xy}\, dy\, dx = \int_0^1 e^{xy}\Big|_0^1 dx = \int_0^1 e^x - 1\, dx = e^x - x\Big|_0^1 = e - 1 - 1 = e - 2$$

2. Compute: $\displaystyle\iint_R 1/x\, dA,\ R = \{(x,y) \mid 1 \le y \le e, y^2 \le x \le y^4\}$.

   Solution:

$$\int_1^e \int_{y^2}^{y^4} \frac{1}{x}\, dx\, dy = \int_1^e \ln y^4 - \ln y^2 dy = \int_1^e 4\ln y - 2\ln y dy = \int_1^e 2\ln y\, dy$$
$$= 2\big[y\ln y - y\big]_1^e = 2(e - e - 0 + 1) = 2$$

3. Compute: Compute: $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$.

   Solution:

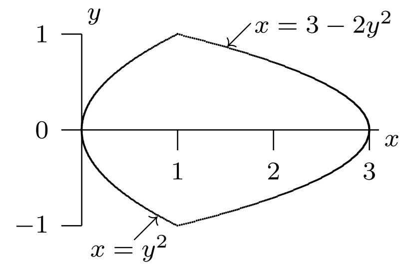
$$\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy = \int_0^1 \int_0^x \sin(x^2)\, dy\, dx = \int_0^1 y\sin(x^2)\Big|_0^x dx$$
$$= \int_0^1 x\sin(x^2)\, dx = \frac{-\cos(x^2)}{2}\Big|_0^1 = \frac{-\cos(1)}{2} + \frac{1}{2}$$

4. Find the volume under $z = x^2 + y^2$ and above $[-2,2]\times[-3,3]$.

   Solution:

$$\int_{-2}^2 \int_{-3}^3 x^2 + y^2\, dy\, dx = \int_{-2}^2 x^2 y + \frac{y^3}{3}\Big|_{-3}^3 dx = \int_{-2}^2 6x^2 + 18\, dx = \frac{6x^3}{3} + 18x\Big|_{-2}^2 = 104$$

5. Set up an integral for the surface area of $z = y^2 - x + 3$ above the region shown. Do not evaluate the integral.



Solution: The partial derivatives are $f_x = 1$, $f_y = 2y$, and the integral is

$$\int_{-1}^{1}\int_{y^2}^{3-2y^2} \sqrt{1^2 + (2y)^2 + 1}\,dx\,dy = \int_{-1}^{1}\int_{y^2}^{3-2y^2} \sqrt{2 + 4y^2}\,dx\,dy.$$

6. Find the volume under $z = xy$ and above the region inside $r = 1 + \cos\theta$ in the first quadrant.

Solution: $\displaystyle\int_0^{\pi/2}\int_0^{1+\cos\theta} \sin\theta\cos\theta r^3\,dr\,d\theta = \int_0^{\pi/2} \frac{(1+\cos\theta)^4}{4}\sin\theta\cos\theta\,d\theta.$

Now use $u = 1 + \cos\theta$, $du = -\sin\theta\,d\theta$:

$$\int_2^1 -\frac{u^4}{4}(u-1)\,du = \frac{1}{4}\int_2^1 -(u^5 - u^4)\,du = \frac{1}{4}\left(\frac{-u^6}{6} + \frac{u^5}{5}\right)\Big|_2^1$$

$$= \frac{1}{4}\left(-\frac{1}{6} + \frac{1}{5} + \frac{2^6}{6} - \frac{2^5}{5}\right) = \frac{43}{40}$$

7. A flat plate has the shape bounded by the parabola $y = 9 - x^2$ and the x-axis; the density is given by $\sigma(x, y) = x^2 y$. Set up the three integrals required to compute the center of mass; do not evaluate the integrals.

Solution:

$$M = \int_{-3}^{3}\int_0^{9-x^2} x^2 y\,dy\,dx$$

$$M_x = \int_{-3}^{3}\int_0^{9-x^2} x^2 y^2\,dy\,dx$$

$$M_y = \int_{-3}^{3}\int_0^{9-x^2} x^3 y\,dy\,dx$$

8. Compute: $\int_0^2 \int_0^{\sqrt{9-x^2}} \int_0^{x^2} yz\,dy\,dz\,dx.$

Solution:

$$\int_0^2 \int_0^{\sqrt{9-x^2}} \int_0^{x^2} yz\,dy\,dz\,dx = \frac{1}{2}\int_0^2 \int_0^{\sqrt{9-x^2}} x^4 z\,dz\,dx = \frac{1}{4}\int_0^2 x^4(9-x^2)\,dx$$

$$= \frac{1}{4}\int_0^2 9x^4 - x^6\,dx = \frac{1}{4}\left(\frac{9}{5}x^5 - \frac{x^7}{7}\right)\Big|_0^2 = \frac{9\cdot 2^5}{4\cdot 5} - \frac{2^7}{4\cdot 7} = \frac{344}{35}$$

9. Compute: $\iiint\limits_R x^3 + xy^2\,dV$, where R is the three dimensional region in the first octant that is under $z = 1 - x^2 - y^2$.

Solution: We use cylindrical coordinates. First we modify the integrand:

$$x^3 + xy^2 = r^3\cos^3\theta + r^3\cos\theta\sin^2\theta = r^3\cos\theta(\cos^2\theta + \sin^2\theta) = r^3\cos\theta.$$

Then

$$\int_0^{\pi/2}\int_0^1\int_0^{1-r^2} \cos\theta\, r^4\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 \cos\theta\, r^4(1-r^2)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta\left[\frac{r^5}{5} - \frac{r^7}{7}\right]_0^1 d\theta = \frac{2}{35}\sin\theta\Big|_0^{\pi/2} = \frac{2}{35}$$

10. Find the mass of a hemisphere of radius 1 if the density is $\sigma(x,y,z) = z$, assuming that the sphere is centered at the origin and the hemisphere is the upper half.

Solution: $\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/2} \cos\phi\sin\phi\,d\phi\,d\theta =$
$\frac{1}{4}\int_0^{2\pi} \frac{1}{2}\left[\sin^2\phi\right]_0^{\pi/2} d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4}$