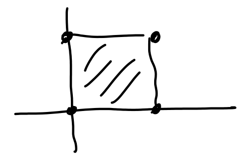
Calculus III—Math 225

SAMPLE EXAM 3

1. Compute: $\displaystyle\iint_R xe^{xy}\,dA$, where R is the square with corners $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$.



$$\int_0^1\int_0^1 x e^{xy}\,dy\,dx = \int_0^1\int_0^x e^u\,du\,dx =$$

$u = xy$ $\quad y=0,\ u=0$

$du = x\,dy$ $\quad y=1,\ u=x$

$$\int_0^1 e^u\Big|_0^x\,dx = \int_0^1 e^x - 1\,dx$$

$$= e^x - x\Big|_0^1 = e^1 - 1 - (1) = e-2$$

2. Compute: $\displaystyle\iint_R 1/x\,dA$, $R = \{(x,y) \mid 1 \le y \le e, y^2 \le x \le y^4\}$.

$$\int_1^e\int_{y^2}^{y^4} \frac{1}{x}\,dx\,dy = \int_1^e \ln x\Big|_{y^2}^{y^4} dy = \int_1^e \ln y^4 - \ln y^2\,dy$$

$$= \int_1^e 4\ln y - 2\ln y\,dy = \int_1^e 2\ln y\,dy = 2y\ln y - 2y\Big|_1^e$$

$u = \ln y$ $\quad dv = 2\,dy$

$du = \frac{1}{y}\,dy$ $\quad v = 2y$

$$= 2e\ln e - 2e - 0 + 2$$

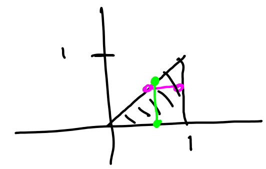
$$= 2e - 2e + 2 = 2$$

$$2y\ln y - \int 2y\frac{1}{y}\,dy = 2y\ln y - \int 2\,dy$$

$$= 2y\ln y - 2y$$

3. Compute: $\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy$.

$x = y$ to $x = 1$



$$\int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \int_0^1 \sin(x^2)\,y \Big|_0^x \,dx$$

$$= \int_0^1 x \sin(x^2)\,dx = \int_0^1 \frac{1}{2} \sin u\,du = \frac{1}{2}(-\cos u)\Big|_0^1$$

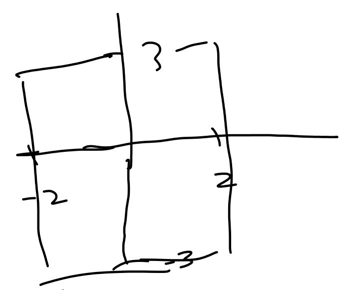
$u = x^2$

$du = 2x\,dx$

$$= \frac{1}{2}(-\cos(1) + 1)$$

4. Find the volume under $z = x^2 + y^2$ and above the rectangle described by $-2 \leq x \leq 2$, $-3 \leq y \leq 3$.

$$\int_{-2}^2 \int_{-3}^3 x^2 + y^2\,dy\,dx = \int_{-2}^2 x^2 y + \frac{y^3}{3} \Big|_{-3}^3 \,dx$$



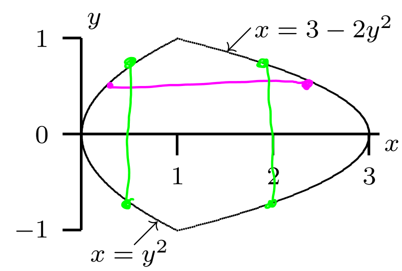
$$= \int_{-2}^2 3x^2 + 9 - (-3x^2 + -9)\,dx$$

$$= \int_{-2}^2 6x^2 + 18\,dx = \frac{6x^3}{3} + 18x \Big|_{-2}^2 = 2 \cdot 2^3 + 18 \cdot 2 - (-16 - 18)$$

$$= 16 + 16 + 36 + 36$$

$$= 72 + 32 = 104$$

5. Set up an integral for the surface area of $z = y^2 - x + 3$ above the region shown. Do not evaluate the integral.

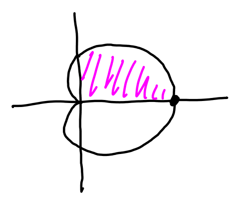


$$\int_{-1}^{1} \int_{y^2}^{3-2y^2} \sqrt{1 + 4y^2 + 1} \; dx\,dy$$

$f_x = 1$

$f_y = 2y$

6. Find the volume under $z = xy$ and above the region inside $r = 1 + \cos\theta$ in the first quadrant.



$$\int_0^{\pi/2} \int_0^{1+\cos\theta} r\cos\theta \; r\sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{1+\cos\theta} \cos\theta\sin\theta \, r^3 \, dr\, d\theta = \int_0^{\pi/2} \cos\theta \sin\theta \left. \frac{r^4}{4} \right|_0^{1+\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \cos\theta \sin\theta \frac{(1+\cos\theta)^4}{4} \, d\theta = -\int_2^1 (u-1) \frac{u^4}{4} \, du$$

$u = 1 + \cos\theta \qquad \cos\theta = u - 1$

$du = -\sin\theta \, d\theta$

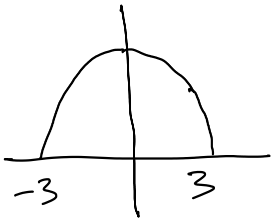
$-du = \sin\theta \, d\theta$

$$= \frac{1}{4}\int_1^2 u^5 - u^4 \, du$$

$$= \frac{1}{4}\left[\frac{u^6}{6} - \frac{u^5}{5} \right]_1^2$$

$$= \frac{1}{4}\left[\frac{2^6}{6} - \frac{2^5}{5} - \frac{1}{6} + \frac{1}{5} \right]$$

7. A flat plate has the shape bounded by the parabola $y = 9 - x^2$ and the x-axis; the density is given by $\sigma(x, y) = x^2 y$. Set up the three integrals required to compute the center of mass; do not evaluate the integrals.



$$M = \int_{-3}^{3}\int_{0}^{9-x^2} x^2 y \, dy \, dx$$

$$M_y = \int_{-3}^{3}\int_{0}^{9-x^2} x^3 y \, dy \, dx$$

$$M_x = \int_{-3}^{3}\int_{0}^{9-x^2} x^2 y^2 \, dy \, dx$$

8. Compute: $\displaystyle\int_0^2 \int_0^{\sqrt{9-x^2}} \int_0^{x^2} yz \, dy \, dz \, dx.$ $= \displaystyle\int_0^2\int_0^{\sqrt{9-x^2}} z \left.\frac{y^2}{2}\right|_0^{x^2} dz\, dx$

$$= \int_0^2\int_0^{\sqrt{9-x^2}} z \frac{x^4}{2} \, dz\, dx = \int_0^2 \frac{x^4}{2} \left.\frac{z^2}{2}\right|_0^{\sqrt{9-x^2}} dx = \int_0^2 \frac{x^4}{2} \frac{9-x^2}{2} \, dx$$

$$= \frac{1}{4}\int_0^2 9x^4 - x^6 \, dx = \frac{1}{4}\left[9\frac{x^5}{5} - \frac{x^7}{7}\right]_0^2$$

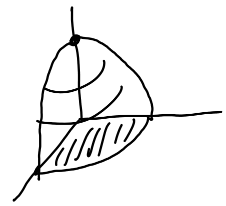
$$= \frac{1}{4}\left[\frac{9}{5}2^5 - \frac{1}{7}2^7\right]$$

9. Compute: $\displaystyle\iiint_R x^3 + xy^2\, dV$, where R is the three dimensional region in the first octant that is under $\underline{z = 1 - x^2 - y^2}$.

$= 1 - r^2$

$k = 1 - x^2 - y^2$

$x^2 + y^2 = 1 - k$



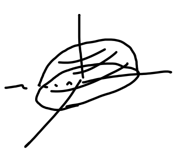
$$\int_0^{\pi/2}\int_0^1\int_0^{1-r^2} \left[(r\cos\theta)^3 + (r\cos\theta)(r\sin\theta)^2\right] r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} \cos\theta\left[\cos^2\theta + \sin^2\theta\right] r^4\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \cos\theta\, r^4\, z\Big|_0^{1-r^2} dr\,d\theta = \int_0^{\pi/2}\int_0^1 \cos\theta\, r^4(1-r^2)\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 \cos\theta\left[r^4 - r^6\right]dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta\left[\frac{r^5}{5} - \frac{r^7}{7}\right]_0^1 d\theta = \int_0^{\pi/2}\left[\frac{1}{5} - \frac{1}{7}\right]\cos\theta\,d\theta = \left[\frac{1}{5} - \frac{1}{7}\right]\sin\theta\Big|_0^{\pi/2} = \frac{1}{5} - \frac{1}{7}$$

10. Find the mass of a hemisphere of radius 1 if the density is $\sigma(x, y, z) = z$, assuming that the sphere is centered at the origin and the hemisphere is the upper half.



$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos\phi \quad \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \cos\phi\sin\phi\ \frac{\rho^4}{4}\Big|_0^1 d\phi\,d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/2} \cos\phi\sin\phi\,d\phi\,d\theta$$

$u = \sin\phi$

$$= \frac{1}{4}\int_0^{2\pi} \frac{\sin^2\phi}{2}\Big|_0^{\pi/2} d\theta = \frac{1}{4}\int_0^{2\pi}\frac{1}{2}\,d\theta = \frac{1}{4}\,\frac{1}{2}\,\theta\Big|_0^{2\pi} = \frac{1}{4}\,\frac{1}{2}\,2\pi = \frac{\pi}{4}$$