Chapter 16 Sample Exam Solutions

1. Compute $\int_C xy\,ds$, where C is given by $\langle 2\sin\theta, 2\cos\theta\rangle$, $0 \le \theta \le \pi/2$.

   Solution:

   $$\mathbf{r}' = \langle 2\cos\theta, -2\sin\theta\rangle,\ |\mathbf{r}'| = 2, \text{ so } \int_C xy\,ds = \int_0^{\pi/2} 2 \cdot 2\sin\theta \cdot 2\cos\theta\,d\theta = 4.$$

2. Explain how you can tell that $\mathbf{F} = \langle 3x^2\cos y, -x^3\sin y\rangle$ is conservative. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is $\langle \cos t, t^2\rangle$, $0 \le t \le 1$.

   Solution:

   $\mathbf{F}$ is conservative because

   $$Q_x = -3x^2\sin y = P_y.$$

   To find f so that $\mathbf{F} = \nabla f$, we compute

   $$\int 3x^2\cos y\,dx = x^3\cos y + C(y)$$
   $$\int -x^3\sin y\,dy = x^3\cos y + D(x)$$

   which match with $C(y) = D(x) = 0$, so $f = x^3\cos y$. Then

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = x^3\cos y\Big|_{(1,0)}^{(\cos(1),1)} = \cos^4(1) - 1.$$

3. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle x^2y^2, 3x + xy\rangle$ and C is the square $(0,0) \to (1,0) \to (1,1) \to (0,1) \to (0,0)$.

   Solution:

   Since the curve C is closed, we may use Green's Theorem. $Q_x = 3 + y$ and $P_y = 2x^2y$, so

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D Q_x - P_y\,dA = \int_0^1\int_0^1 3 + y - 2x^2y\,dy\,dx = 3 + \frac{1}{2} - \frac{1}{3} = \frac{19}{6}.$$

4. Convert $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds$, to a double integral that is ready to evaluate, including the limits, but do not evaluate the integral. The curve C is the circle $x^2 + y^2 = 1$ and $\mathbf{F} = \langle ax^2, by^2 \rangle$.

Solution:

Answer. Using one form of Green's Theorem,

$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2ax + 2by\, dy\, dx.$$

5. Compute $\nabla \times \mathbf{F}$, $\mathbf{F} = \langle x \cos z, y \cos z, \sin z \rangle$. Is $\mathbf{F}$ conservative? Why or why not?

Solution:

$\nabla \times \mathbf{F} = \langle y \sin z, -x \sin z, 0 \rangle$. $\mathbf{F}$ is not conservative because $\nabla \times \mathbf{F}$ is not the zero vector.

6. Compute $\nabla \cdot \mathbf{F}$, $\mathbf{F} = \langle x \cos z, y \cos z, \sin z \rangle$.

Solution:

$\nabla \cdot \mathbf{F} = \cos z + \cos z + \cos z = 3 \cos z$.

7. Set up a double integral for the surface area of $\mathbf{r} = \langle u^2, u^2 - v, v^3 \rangle$, $0 \le u \le 1$, $0 \le v \le 1$.

Solution:

$\mathbf{r}_u = \langle 2u, 2u, 0 \rangle$, $\mathbf{r}_v = \langle 0, -1, 3v^2 \rangle$, $\mathbf{r}_u \times \mathbf{r}_v = \langle 6uv^2, -6uv^2, -2u \rangle$, and $|\mathbf{r}_u \times \mathbf{r}_v| = \left(36u^2v^4 + 36u^2v^4 + 4u^2\right)^{1/2}$. The surface area is then

$$\int_0^1 \int_0^1 \left(36u^2v^4 + 36u^2v^4 + 4u^2\right)^{1/2}\, du\, dv.$$

8. Compute $\iint_D \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F} = \langle y, z, x\rangle$ and D is the surface $z = x^2 + y^2$ above the interior of the square with corners $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, oriented up.

Solution:

Use $\mathbf{r} = \langle x, y, x^2+y^2\rangle$, so $\mathbf{r}_x = \langle 1,0,2x\rangle$, $\mathbf{r}_y = \langle 0,1,2y\rangle$, and $\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, 1\rangle$. Then

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \int_0^1\int_0^1 \langle y, x^2+y^2, x\rangle\cdot\langle -2x,-2y,1\rangle\,dy\,dx$$

$$= \int_0^1\int_0^1 -2xy - 2x^2y - 2y^3 + x\,dy\,dx = -5/6.$$

9. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle z^2, y, x\rangle$ and C is the triangle $(1,0,0)\to(0,1,0)\to(0,0,1)\to(1,0,0)$.

Solution:

Since the curve C is closed we may use Stokes's Theorem. The curve is the boundary of the surface D given by $\mathbf{r} = \langle x, y, 1-x-y\rangle$, $0\le x\le 1$, $0\le y\le 1-x$, a triangular portion of the plane $z = 1-x-y$. Now $\mathbf{r}_x = \langle 1,0,-1\rangle$, $\mathbf{r}_y = \langle 0,1,-1\rangle$, and $\mathbf{r}_x\times\mathbf{r}_y = \langle 1,1,1\rangle$. Also $\nabla\times\mathbf{F} = \langle 0, 2z-1, 0\rangle$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{N}\,dS = \iint_D \langle 0,2z-1,0\rangle\cdot\langle 1,1,1\rangle\,dA$$

$$= \int_0^1\int_0^{1-x} 2(1-x-y) - 1\,dy\,dx = -1/6.$$

10. Compute $\iint_D \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F} = \langle x^2z, z^2y, y^2x\rangle$ and D is the surface of the cube with corners $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(0,0,1)$, $(0,1,0)$, $(1,1,0)$, $(1,1,1)$, $(0,1,1)$, oriented outward.

Solution:

The surface is closed so we use the divergence theorem. $\nabla\cdot\mathbf{F} = 2xz + z^2$ and then

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \int_0^1\int_0^1\int_0^1 2xz + z^2\,dy\,dx\,dz = 5/6.$$