Chapter 16 Sample Exam Solutions

1. Compute $\int_C xy\,ds$, where C is given by $\langle 2\sin\theta, 2\cos\theta\rangle$, $0 \le \theta \le \pi/2$.

$\bar{r} = \langle 2\sin\theta, 2\cos\theta \rangle$

$\bar{r}' = \langle 2\cos\theta, -2\sin\theta \rangle$

$|\bar{r}'| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = 2$

$ds = |\bar{r}'|d\theta$

$$\int_0^{\pi/2} 2\sin\theta\, 2\cos\theta\, 2\, d\theta = 8\,\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = 8\cdot\frac{1}{2} = 4$$

2. Explain how you can tell that $\mathbf{F} = \langle \overset{P}{3x^2\cos y}, \overset{Q}{-x^3\sin y}\rangle$ is conservative. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is $\langle \cos t, t^2\rangle$, $0 \le t \le 1$.

$Q_x = -3x^2\sin y$, $P_y = -3x^2\sin y$, $Q_x = P_y$, so F is conservative.

$$\int_C \bar{F}\cdot d\bar{r} = f\Big|_{(1,0)}^{(\cos(1),1)} = x^3\cos y\Big|_{(1,0)}^{(\cos(1),1)} = \cos^3(1)\cos(1) - 1\cdot 1$$

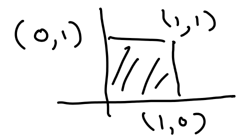
$$= \cos^4(1) - 1$$

$f = \int P\,dx = \int 3x^2\cos y\,dx = x^3\cos y + \underline{\;0\;}$

$f = \int Q\,dy = \int -x^3\sin y\,dy = x^3\cos y + \underline{\;0\;}$

3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle \overset{P}{x^2y^2}, \overset{Q}{3x + xy} \rangle$ and C is the square $(0,0) \to (1,0) \to (1,1) \to (0,1) \to (0,0)$.

Use Green's Theorem:

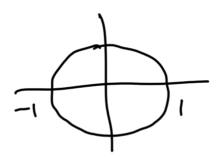


$$\int_0^1\int_0^1 Q_x - P_y \, dy\,dx = \int_0^1\int_0^1 3+y-2x^2y \, dy\,dx$$

$$= \int_0^1 3y + \frac{y^2}{2} - 2x^2\frac{y^2}{2}\Big|_0^1 dx = \int_0^1 3+\frac{1}{2} - x^2 \, dx$$

$$= 3x + \frac{x}{2} - \frac{x^3}{3}\Big|_0^1 = 3 + \frac{1}{2} - \frac{1}{3}.$$

4. Convert $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds$, to a double integral that is ready to evaluate, including the limits, but do not evaluate the integral. The curve C is the circle $x^2 + y^2 = 1$ and $\mathbf{F} = \langle ax^2, by^2 \rangle$.



$$\oint_C \bar{F}\cdot\bar{N}\,ds = \iint_D \nabla\cdot F\,dA = \iint_D 2ax+2by\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2ax+2by\,dy\,dx$$

5. Compute $\nabla \times \mathbf{F}$, $\mathbf{F} = \langle x\cos z, y\cos z, \sin z\rangle$. Is $\mathbf{F}$ conservative? Why or why not?

$$\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$$
$$\times \langle x\cos z, y\cos z, \sin z \rangle$$
$$= \langle 0 - y(-\sin z), -x\sin z - 0, 0 - 0 \rangle$$
$$= \langle y\sin z, -x\sin z, 0 \rangle$$

$\bar{F}$ is not conservative because $\nabla \times \bar{F} \neq \vec{0}$

6. Compute $\nabla \cdot \mathbf{F}$, $\mathbf{F} = \langle x\cos z, y\cos z, \sin z\rangle$.

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x\cos z, y\cos z, \sin z \rangle$$
$$= \cos z + \cos z + \cos z = 3\cos z$$

7. Set up a double integral for the surface area of $\mathbf{r} = \langle u^2, u^2 - v, v^3\rangle$, $0 \le u \le 1$, $0 \le v \le 1$.

$\bar{r} = \langle u^2, u^2 - v, v^3 \rangle$ Surf. area $= \iint_D |\bar{r}_u \times \bar{r}_v| \, du\,dv$

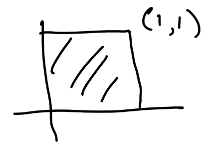
$\bar{r}_u = \langle 2u, 2u, 0 \rangle$

$\bar{r}_v = \langle 0, -1, 3v^2 \rangle$

$\bar{r}_u \times \bar{r}_v = \langle 6uv^2, -6uv^2, -2u \rangle$

$$\text{S.A.} = \int_0^1 \int_0^1 \sqrt{36u^2v^4 + 36u^2v^4 + 4u^2} \; du\,dv$$

8. Compute $\displaystyle\iint_D \mathbf{F} \cdot \mathbf{N}\, dS$, where $\mathbf{F} = \langle y, z, x\rangle$ and D is the surface $z = x^2 + y^2$ above the interior of the square with corners $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, oriented up.



$\bar{r} = \langle u, v, u^2 + v^2 \rangle$

$\bar{r}_u = \langle 1, 0, 2u \rangle$

$\bar{r}_v = \langle 0, 1, 2v \rangle$

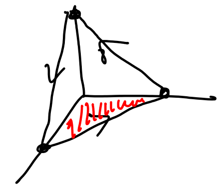
$\bar{r}_u \times \bar{r}_v = \langle -2u, -2v, 1 \rangle$

$$\int_0^1 \int_0^1 \langle v, u^2 + v^2, u \rangle \cdot \langle -2u, -2v, 1 \rangle \, dv\,du = \int_0^1 \int_0^1 -2uv - 2vu^2 - 2v^3 + u \, dv\,du$$

$$= \int_0^1 -uv^2 - v^2u^2 - 2\frac{v^4}{4} + uv \Big|_0^1 du = \int_0^1 -u - u^2 - \frac{1}{2} + u \, du$$

$$= -\frac{u^2}{2} - \frac{u^3}{3} - \frac{1}{2}u + \frac{u^2}{2} \Big|_0^1 = -\frac{1}{2} - \frac{1}{3} - \frac{1}{2} + \frac{1}{2} = -\frac{1}{2} - \frac{1}{3} = \frac{-5}{6}$$

9. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle z^2, y, x\rangle$ and C is the triangle $(1,0,0) \to (0,1,0) \to (0,0,1) \to (1,0,0)$.



Stokes's Theorem

$\bar{r} = \langle u, v, \underline{1-u-v} \rangle$

$\bar{r}_u = \langle 1, 0, -1 \rangle$

$\bar{r}_v = \langle 0, 1, -1 \rangle$

$\bar{r}_u \times \bar{r}_v = \langle 1, 1, 1 \rangle$

$\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$

$\times \langle z^2, y, x \rangle$

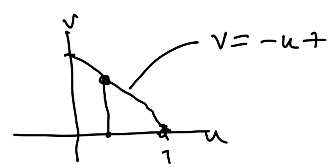
$= \langle 0-0, 2z-1, 0-0 \rangle$

$= \langle 0, 2z-1, 0 \rangle$

$x+y+z=1$

$z = 1-x-y$

$$\int_C \bar{F}\cdot d\bar{r} = \iint_D (\nabla \times \bar{F}) \cdot \langle \bar{r}_u \times \bar{r}_v \rangle \, dv\, du = \iint_D \langle 0, 2(1-u-v)-1, 0 \rangle \cdot \langle 1, 1, 1 \rangle \, dv\, du$$



$$= \int_0^1 \int_0^{1-u} 2 - 2u - 2v - 1 \, dv\, du$$

$$= \int_0^1 v - 2uv - v^2 \Big|_0^{1-u} du$$

$$= \int_0^1 (1-u) - 2u(1-u) - (1-u)^2 \, du$$

$$= \int_0^1 1 - u - 2u + 2u^2 - (1 - 2u + u^2)\, du$$

$$= \int_0^1 \cancel{1} - \cancel{3u} + \cancel{2u^2} - \cancel{1} + \cancel{2u} - \cancel{u^2} \, du$$

$$= \int_0^1 -u + u^2 \, du = -\frac{u^2}{2} + \frac{u^3}{3} \Big|_0^1 = -\frac{1}{2} + \frac{1}{3} = \frac{-1}{6}$$

10. Compute $\iint\limits_D \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F} = \langle x^2z, z^2y, y^2x\rangle$ and D is the surface of the cube with corners $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(0,0,1)$, $(0,1,0)$, $(1,1,0)$, $(1,1,1)$, $(0,1,1)$, oriented outward.

$$\iint\limits_D \bar{F}\cdot\bar{N}\,dS = \int_0^1\int_0^1\int_0^1 2xz + z^2 + 0 \; dz\,dy\,dx$$

$$\nabla\cdot\bar{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle x^2z, z^2y, y^2x\rangle$$

$$= \int_0^1\int_0^1 xz^2 + \frac{z^3}{3}\Big|_0^1 dy\,dx = \int_0^1\int_0^1 x + \frac{1}{3}\,dy\,dx$$

$$= \int_0^1 xy + \frac{y}{3}\Big|_0^1 dx = \int_0^1 x + \frac{1}{3}\,dx = \frac{x^2}{2} + \frac{x}{3}\Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$