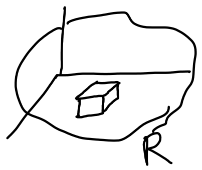
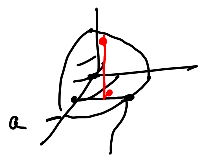
Triple integrals, sections 15.5, 15.6

Triple Integrals



$$\Delta x \Delta y \Delta z \longrightarrow \iiint_R dx\,dy\,dz$$

$x^2+y^2+z^2=a^2$, first octant:

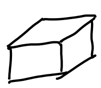


$y=\sqrt{a^2-x^2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} 1\,dz\,dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} z\Big|_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

Total mass, density yz.



Vol: $\Delta x \Delta y \Delta z$ } mass: $yz\,\Delta x \Delta y \Delta z$
density yz

$$\text{Total mass} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} yz\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} y\frac{z^2}{2}\Big|_0^{\sqrt{a^2-x^2-y^2}} dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{y(a^2-x^2-y^2)}{2}\,dy\,dx$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{M_{xz}}{M}$$

$$\bar{z} = \frac{M_{xy}}{M}$$

$$\bar{x} = \iiint_R x\,\sigma(x,y,z)\,dz\,dx\,dy$$

$$\bar{y} = \iiint_R y\,\sigma(xyz)\,dz\,dy\,dx$$

$$\bar{z} = \iiint_R z\,\sigma(xyz)\,dz\,dy\,dx$$

Average value of a function: $f(x,y)$ = temperature at point (x,y), average temperature over region R:

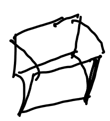
$$\frac{1}{A}\iint_R f(x,y)\,dy\,dx$$

<u>3D</u> $f(x,y,z)$ = temp. at (x,y,z), average temp:

$$\frac{1}{V}\iiint_R f(x,y,z)\,dz\,dx\,dy$$

Cylindrical coordinates: (r,θ,z)

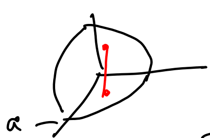
$$\iiint_R f(x,y,z)\; r\,dr\,d\theta\,dz$$



$r\,\Delta r\,\Delta\theta\,\Delta z$

$$\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} f(x,y,z)\; r\,dz\,dr\,d\theta$$

density $= yz$
$= r\sin\theta\, z$



$z=\sqrt{a^2-r^2}$

Total mass: $\displaystyle\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} r\sin\theta\, z\; r\,dz\,dr\,d\theta$

$$\int_0^{\pi/2}\int_0^a r^2\sin\theta\,\frac{z^2}{2}\Big|_0^{\sqrt{a^2-r^2}} dr\,d\theta = \int_0^{\pi/2}\int_0^a r^2\sin\theta\,\frac{(a^2-r^2)}{2}\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^a a^2r^2\sin\theta - r^4\sin\theta\,dr\,d\theta = \frac{1}{2}\int_0^{\pi} a^2\frac{r^3}{3}\sin\theta - \frac{r^5}{5}\sin\theta\Big|_0^a d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{a^5}{3}-\frac{a^5}{5}\right)\sin\theta\,d\theta = \frac{1}{2}\left(\frac{a^5}{3}-\frac{a^5}{5}\right)(-\cos\theta)\Big|_0^{\pi/2} = \frac{1}{2}\left(\frac{a^5}{3}-\frac{a^5}{5}\right)(1)$$