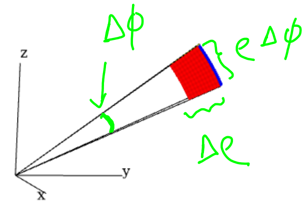
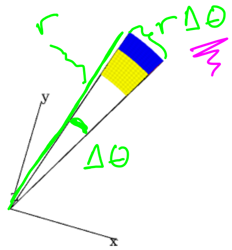
Triple integrals in spherical coordinates, section 15.6

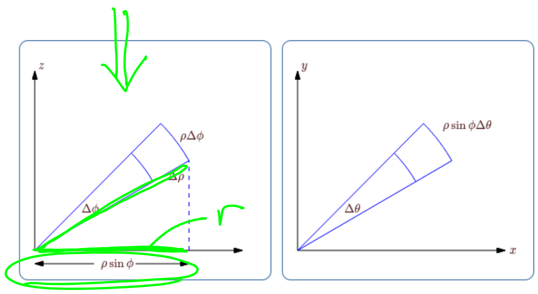
$\Delta\rho \, \Delta\phi \, \Delta\theta$

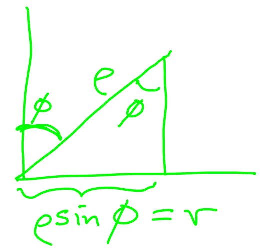




Little box has volume

$$\underbrace{\Delta\rho}\ \underbrace{\rho\,\Delta\phi}\ \underbrace{\rho\sin\phi\,\Delta\theta}$$





$$\text{Volume} = \iiint_R 1 \quad \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Volume of a sphere of radius a.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\Big|_0^a \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^3}{3} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \frac{a^3}{3}(-\cos\phi)\Big|_0^{\pi} d\theta = \int_0^{2\pi} \frac{a^3}{3}(--1--1) \, d\theta$$

$$= \frac{a^3}{3} 2 \int_0^{2\pi} d\theta = \frac{a^3}{3} \cdot 2 \ \theta \Big|_0^{2\pi} = \frac{a^3}{3} \cdot 2 \cdot 2\pi = \underline{\underline{\frac{4}{3}\pi a^3}}$$

$$\iiint_R f(\rho,\phi,\theta) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$f(x,y,z)$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

density $= z^2$

Total mass $= \displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{a} (\rho\cos\phi)^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^4 \underbrace{\cos^2\phi \sin\phi}\, d\rho \, d\phi \, d\theta$$

$\displaystyle\int \underbrace{\cos^2\phi}_{u^2} \overbrace{\sin\phi \, d\phi}^{-du}$

$u = \cos\phi$

$du = -\sin\phi \, d\phi$

$$M_{yz} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \underbrace{\rho\sin\phi\cos\theta}_{x} \, \rho^4 \cos^2\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

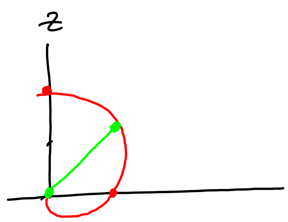
$\bar{x} = \dfrac{M_{yz}}{M}$

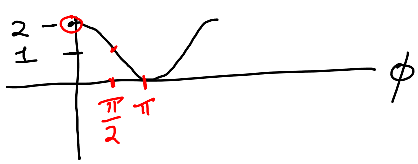
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^5 \sin^2\phi \cos^2\phi \cos\theta \, d\rho \, d\phi \, d\theta$$

$\displaystyle\int \sin^2\phi \cos^2\phi \, d\phi$??

$\rho = \underbrace{1+\cos\phi}$

"Same" in every θ-direction.





$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \sin\phi \left.\frac{\rho^3}{3}\right|_0^{1+\cos\phi} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\phi \, \frac{(1+\cos\phi)^3}{3} \, d\phi \, d\theta = \int_0^{2\pi}\int_2^{0} -\frac{u^3}{3} \, du \, d\theta$$

$u = 1+\cos\phi$

$du = -\sin\phi \, d\phi$

$-du = \sin\phi \, d\phi$

$u = 1+\cos 0 = 2$

$u = 1+\cos\pi = 1-1 = 0$

$$= \int_0^{2\pi}\int_0^2 \frac{u^3}{3}\,du\,d\theta = \int_0^{2\pi} \frac{u^4}{12}\Big|_0^2 d\theta = \int_0^{2\pi} \frac{2^4}{12}\,d\theta$$

$$= \frac{2^4}{12}\,\theta\Big|_0^{2\pi} = \frac{2^4}{12}2\pi = \frac{4}{3}\cdot 2\pi = \frac{8\pi}{3}$$