Change of variables, section 15.7

$x = \sin u$

$b = \sin u$

$\arcsin b = u$

$$\int_a^b f(x)\,\underbrace{dx} \longrightarrow \int_{-}^{-} f(\sin u)\underbrace{\cos u\, du}$$

$dx = \cos u\, du$

— o —

$$\iint h(x,y)\,dy\,dx \longrightarrow \iint H(u,v)\,\underbrace{C(u,v)}\,du\,dv$$

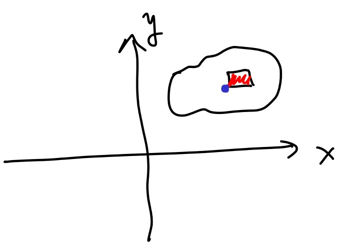
$x = f(u,v)$

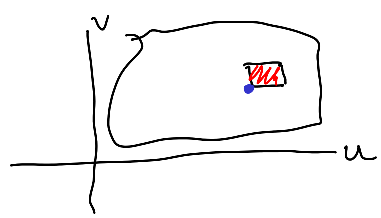
$y = g(u,v)$

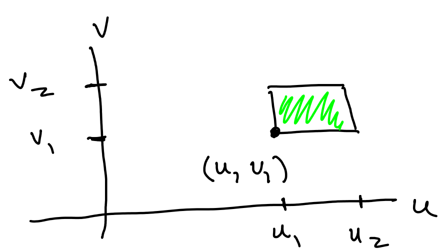
$H(u,v) = h(f(u,v), g(u,v))$

1. Find new limits.
2. Find $H(u,v)$ — easy
3. Find the correction factor $C(u,v)$

$$\iint h(x,y)\,dy\,dx \qquad \iint H(u,v)\,C(u,v)\,du\,dv$$







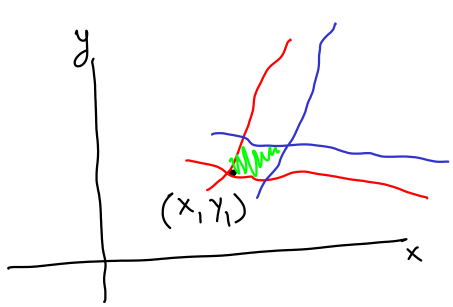
☆ $u = u_1$ $\langle f(u_1, v), g(u_1, v), 0 \rangle$

vector function for a curve.

☆ $u = u_2$ $\langle f(u_2, v), g(u_2, v), 0 \rangle$

☆ $v = v_1$ $\langle f(u, v_1), g(u, v_1), 0 \rangle$

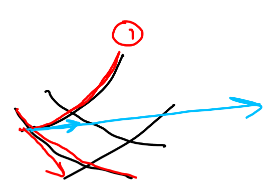
☆ $v = v_2$ $\langle f(u, v_2), g(u, v_2), 0 \rangle$



$x_1 = f(u_1, v_1)$

$y_1 = g(u_1, v_1)$

In xy plane:



$$\langle f(u,v), g(u,v), 0\rangle \xrightarrow[\text{deriv.}]{} \langle f_v(u,v), g_v(u,v), 0\rangle$$

Vector: $\langle f_v(u,v_1), g_v(u,v_1), 0\rangle \Delta v$

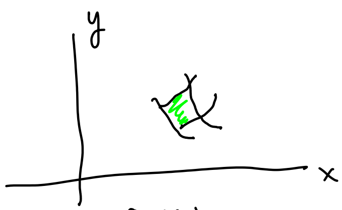
$$\langle f(u,v_1), g(u,v_1), 0\rangle \longrightarrow \langle f_u(u,v_1), g_u(u,v_1), 0\rangle$$

Vector: $\langle f_u(u,v_1), g_u(u,v_1), 0\rangle \Delta u$

$$\begin{array}{r} \langle f_v, g_v, 0\rangle \\ \times\ \langle f_u, g_u, 0\rangle \\ \hline \langle 0, 0, f_v g_u - f_u g_v\rangle \end{array}$$

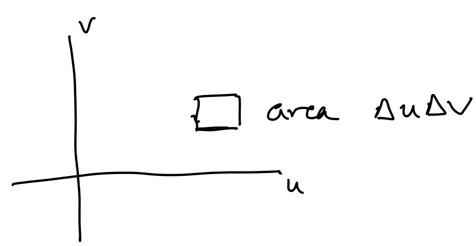
Area: $|\langle 0, 0, f_v g_u - f_u g_v\rangle| \Delta u \Delta v$

$$= |f_v g_u - f_u g_v| \Delta u \Delta v$$



area:

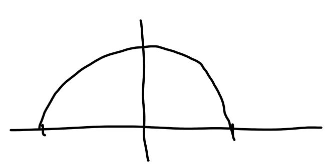
$|f_v g_u - f_u g_v| \Delta u \Delta v$



$$\iint h(x,y)\,dy\,dx = \iint H(u,v)\,\underbrace{|f_v g_u - f_u g_v|}_{}\,du\,dv$$

area = area in xy plane

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$



$f = x = u\cos v = r\cos\theta$

$g = y = u\sin v = r\sin\theta$

$f_r = \cos\theta \qquad g_r = \sin\theta$

$f_\theta = r\sin\theta \qquad g_\theta = -r\cos\theta$

$x^2 + y^2 = 1$

$r^2\cos^2\theta + r^2\sin^2\theta = 1$

$r^2 = 1$

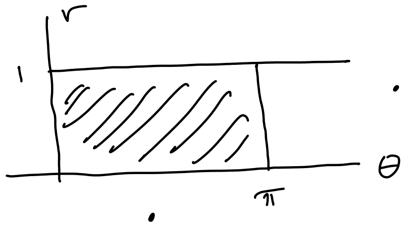
$r = 1$

$y = 0$

$r\sin\theta = 0$

$r = 0$ or $\sin\theta = 0$

$\theta = 0$ or π



$$\left|f_r g_\theta - f_\theta g_r\right|$$
$$= \left|-r\cos^2\theta - r\sin^2\theta\right|$$
$$= |-r| = r$$

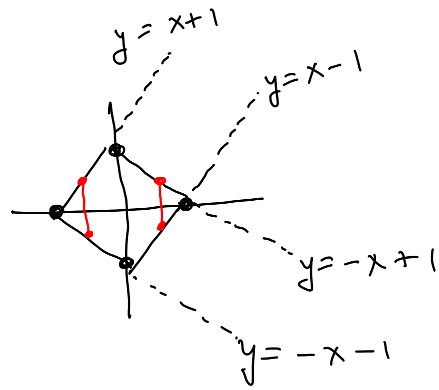
$$\int_0^\pi \int_0^1 r \qquad r\,dr\,d\theta$$

$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$\sqrt{r^2} = r$$

$$= \int_0^\pi \int_0^1 r^2\,dr\,d\theta$$

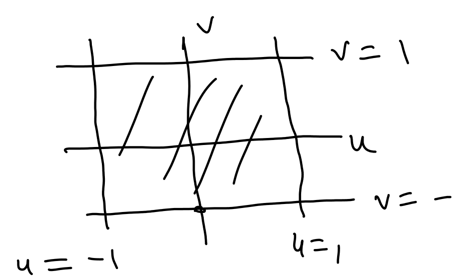
Ex 15.7.3 Evaluate $\iint x^2 + y^2\,dx\,dy$ over the square with corners $(-1,0)$, $(0,1)$, $(1,0)$, and $(0,-1)$ in two ways: directly, and using $x = \underbrace{(u+v)/2}_{f}$, $y = \underbrace{(u-v)/2}_{g}$. (answer)

$$x = \frac{u}{2} + \frac{v}{2}$$
$$y = \frac{u}{2} - \frac{v}{2}$$



$$f_u = \frac{1}{2} \qquad g_u = \frac{1}{2}$$
$$f_v = \frac{1}{2} \qquad g_v = -\frac{1}{2}$$

Jacobian: $\left|-\frac{1}{4} - \frac{1}{4}\right| = \frac{1}{2}$



$y = x+1$: $\cancel{\frac{u}{2}} - \frac{v}{2} = \cancel{\frac{u}{2}} + \frac{v}{2} + 1$

$-v = 1$, $v = -1$

$y = x-1$: $\cancel{\frac{u}{2}} - \frac{v}{2} = \cancel{\frac{u}{2}} + \frac{v}{2} - 1$

$-v = -1$, $v = 1$

$y = -x+1$: $\frac{u}{2} - \cancel{\frac{v}{2}} = -\frac{u}{2} - \cancel{\frac{v}{2}} + 1$

$u = 1$

$y = -x-1$: $\frac{u}{2} - \cancel{\frac{v}{2}} = -\frac{u}{2} - \cancel{\frac{v}{2}} - 1$

$u = -1$

$$\iint_R x^2+y^2\,dA = \int_{-1}^1 \int_{-1}^1 \left[\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right]\frac{1}{2}\,dv\,du$$

$$= \frac{1}{8}\int_{-1}^1 \int_{-1}^1 u^2 + 2uv + v^2 + u^2 - 2uv + v^2\,dv\,du$$

$$= \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1} 2u^2+2v^2 \, dv\, du = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} u^2+v^2 \, dv\, du$$

$$= \frac{1}{4}\int_{-1}^{1} u^2 v + \frac{v3}{3}\Big|_{-1}^{1} du = \frac{1}{4}\int_{-1}^{1} u^2 + \frac{1}{3} - \left(-u^2 - \frac{1}{3}\right) du$$

$$= \frac{1}{4}\int_{-1}^{1} 2u^2 + \frac{2}{3}\, du = \frac{1}{4}\left[\frac{2}{3}u^3 + \frac{2}{3}u\right]_{-1}^{1} = \frac{1}{4}\left[\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right] = \frac{2}{3}$$