Line Integrals, section 16.2

$\left[\begin{array}{l} \int_a^b f(x)\,dx \quad \text{integral over interval or line segment of } f(x). \\ \int_a^b \int_c^d f(x,y)\,dy\,dx \quad \text{integral over a plane region.} \end{array} \right.$

$\iiint_R f(x,y,z)\,dV$ integral over 3D region.

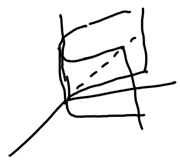
$\int_a^b \langle f, g, h\rangle\,dt$

$r(t) = \langle f, g, h\rangle$

Integral "along a curve."

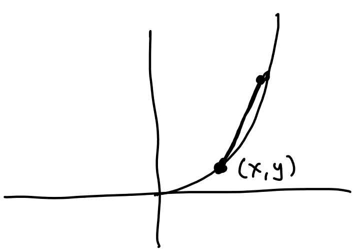
Line integral — "curve integral". Integral over a curve instead of a line segment along an axis.

The surface $y = x^2$



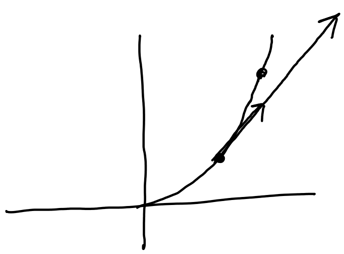
We want the area of part of this surface.

$z = x + y$: a plane. What is the area of $y = x^2$, above xy-plane, below the plane $z = x + y$, and above some section of $y = x^2$.



area of rectangle = length of base · height up to $z = x + y$.

Represent $y = x^2$ as $\bar{r}(t) = \langle t, t^2, 0\rangle$



$\bar{r}'(t) = \langle 1, 2t\rangle$

Small tangent vector: $\langle 1, 2t\rangle \Delta t$

Length: $|\langle 1, 2t\rangle|\,\Delta t = |\bar{r}'|\,\Delta t$

$\left[\int_a^b |\bar{r}'|\,dt = \text{arclength} \right]$

Rectangle area: $(x+y)|\langle 1, 2t\rangle|\,\Delta t$

Area of surface: $\int_a^b (x+y)\,|\langle 1, 2t\rangle|\,dt = \int f(x,y)\,|\bar{r}'|\,dt$

$\bar{r}(t) = \langle t, t^2\rangle$

$$\int_0^2 (t+t^2)\sqrt{1+4t^2}\,dt = \int_0^2 t\sqrt{1+4t^2} + t^2\sqrt{1+4t^2}\,dt$$

"Integral of $f(x,y) = x+y$ along the curve $y=x^2$ from $x=0$ to $x=2$."

Example Integrate ye^x along the straight line from $(1,2)$ to $(4,7)$.

$\bar{r}(t) = \langle 1,2\rangle + t\langle 3,5\rangle = \langle 1+3t, 2+5t\rangle$

$\bar{r}'(t) = \langle 3,5\rangle \quad |\bar{r}'| = \sqrt{9+25} = \sqrt{34}$

$$\int_0^1 (2+5t)e^{1+3t}\sqrt{34}\,dt = \sqrt{34}\int_0^1 2e^{1+3t} + 5te^{1+3t}\,dt$$

$\underbrace{|\bar{r}'|dt}$ $\approx$ length of a short piece of the curve: ds

$\int_0^1 f(x,y)\,ds$ If $\bar{r}(t)$ describes curve C:

$= \int_C f\,ds$

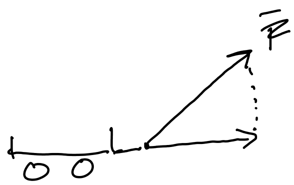
Example $\int_C x^2 z\,ds$ along C: line segment from $(0,6,-1)$ to $(4,1,5)$

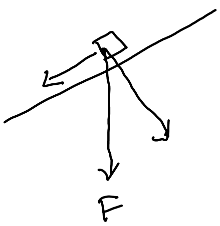
$\bar{r}(t) = \langle 0,6,-1\rangle + t\langle 4,-5,6\rangle = \langle 4t, 6-5t, -1+6t\rangle$

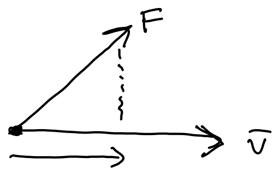
$\bar{r}'(t) = \langle 4,-5,6\rangle$

$$\int_0^1 (4t)^2(6t-1)\sqrt{16+25+36}\,dt$$

Work : force · distance. $W = 10\cdot 5 = 50$ foot-lbs







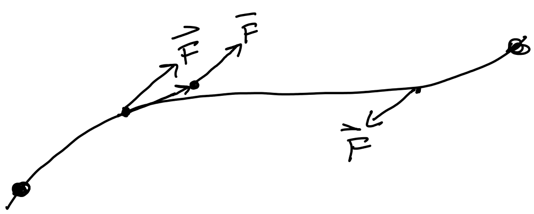
How much work?

Scalar force in direction $\bar{v}$:

$$\frac{\vec{F}\cdot\bar{v}}{|\bar{v}|}$$

$$\text{Work} = \frac{\bar{F}\cdot\bar{v}}{|\bar{v}|}|\bar{v}| = \bar{F}\cdot\bar{v}$$

$\bar{F}$ is a vector field. An object moves through the field.



$\bar{r}(t)$, $\bar{r}'(t)$ is velocity, $\bar{r}'\Delta t$ is a short tangent vector.

Work approximately: $\vec{F}(t)\cdot\bar{r}'\Delta t$.

Total $\displaystyle\int_a^b \vec{F}(t)\cdot\bar{r}'\,dt$

$\bar{r}'dt$: short tangent vector, written $d\bar{r}$

$$\int_a^b \bar{F}(t)\cdot d\vec{r} \qquad \bar{F}\cdot\bar{r}\,dt = \frac{\bar{F}\cdot\bar{r}'}{|\bar{r}'|}\underbrace{|r'|\,dt}_{ds} \qquad \frac{\bar{r}'}{|r'|} = \vec{T}$$

$$\int_a^b \vec{F}\cdot\vec{T}\,ds \qquad = \vec{F}\cdot\vec{T}\,ds$$

$$\vec{F}(x,y,z) = \langle f, g, h\rangle \qquad F = \Big\langle \underset{f}{\frac{-x}{(x^2+y^2+z^2)^{3/2}}},\ \underset{g}{\cdots},\ \underset{h}{\cdots} \Big\rangle$$

$$\bar{r}' = \langle x', y', z'\rangle$$

$$\bar{F}\cdot\bar{r}'dt = \langle f, g, h\rangle\cdot\langle x', y', z'\rangle\,dt$$

$$= f x' dt + g y' dt + h z' dt \qquad x' = \frac{dx}{dt} \qquad x' dt = \frac{dx}{\cancel{dt}} \cancel{dt}$$

$$= f\,dx + g\,dy + h\,dz$$

$$\int_C \bar{F} \cdot \bar{r}'\,dt = \int_{x_1}^{x_2} f\,dx + \int_{y_1}^{y_2} g\,dy + \int_{z_1}^{z_2} h\,dz$$

<u>Example</u> Object moves along $\vec{r} = \langle t, t^2 \rangle$ from $(-1,1)$ to $(2,4)$ subject to $\vec{F} = \langle x\sin y, y \rangle$.

$$\bar{r}' = \langle 1, 2t \rangle \qquad \bar{F} = \langle t\sin(t^2), t^2 \rangle$$

$$\int_{-1}^{2} \langle t\sin t^2, t^2 \rangle \cdot \langle 1, 2t \rangle\,dt$$

$$= \int_{-1}^{2} \underline{t\sin t^2} + \underline{2t^3}\,dt = \cdots$$

$$\bar{F} = \langle \underset{f}{x\sin y}, \underset{g}{y} \rangle$$

$$\underset{(x,y)}{(-1,1)} \rightarrow \underset{(x,y)}{(2,4)} \qquad \int_{-1}^{2} x\sin y\,dx + \int_{1}^{4} y\,dy$$

$$\int_{-1}^{2} x\sin(x^2)\,dx =$$

$$\int_{1}^{4} y\,dy =$$