The fundamental theorem of line integrals, section 16.3

Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Fundamental Theorem of Line Integrals: Suppose C is a curve, given by $\bar{r}(t)$, as $t: a \to b$.

$\vec{b} = \bar{r}(b)$
$\vec{a} = \bar{r}(a)$

$$\int_C \nabla f \cdot d\bar{r} = f(\vec{b}) - f(\vec{a})$$

Proof $\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$, $d\bar{r} = \langle x', y', z' \rangle dt$.

$\nabla f = \langle f_x, f_y, f_z \rangle$.

$$\int_a^b \nabla f\, d\bar{r} = \int_a^b \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle dt$$

$$= \int_a^b \underbrace{f_x x' + f_y y' + f_z z'}_{\frac{df}{dt}}\, dt$$

$f = f(x,y,z)$
$= f(x(t), y(t), z(t))$

$$= \int_a^b \frac{df}{dt}\, dt = \int_a^b f'(t)\, dt$$

$$= f(x(t), y(t), z(t))\Big|_a^b = f(\underbrace{x(b), y(b), z(b)}_{\bar{r}(b) = \vec{b}}) - f(\underbrace{x(a), y(a), z(a)}_{\bar{r}(a) = \vec{a}})$$

$$= f(x,y,z)\Big|_{\vec{a}}^{\vec{b}}$$

If $\vec{F}$ is a vector field and $\vec{F} = \nabla f$, then we call $\vec{F}$ a conservative vector field.

If $\vec{F}$ is a force field, $\int_C \bar{F} \cdot d\bar{r}$ computes "work".

If $\vec{F}$ is conservative, $\int_C \bar{F} \cdot d\bar{r} = f(\bar{b}) - f(\bar{a})$.

$$\vec{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$f = (x^2+y^2+z^2)^{-1/2}$$

$$f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\, 2x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

f_y, f_z ✓

$$\nabla f = \vec{F}.$$

Compute work to go from $\langle 1,0,0\rangle$ to $\langle 2,1,-1\rangle$ along $\bar{r}(t) = \langle 1+t, t^3, t\cos\pi t\rangle$.

$$\int_C \bar{F}\cdot d\bar{r} = f(2,1,-1) - f(1,0,0)$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{1}{(x^2+y^2+z^2)^{1/2}}\Bigg|_{(1,0,0)}^{(2,1,-1)} = \frac{1}{(4+1+1)^{1/2}} - \frac{1}{(1+0+0)^{1/2}} = \frac{1}{\sqrt{6}} - 1$$

Suppose C starts & ends at the same place.

$$\int_C \bar{F}\cdot d\bar{r} = f\Big|_{(x_0 y_0 z_0)}^{(x_0 y_0 z_0)} = f(x_0,y_0,z_0) - f(x_0,y_0,z_0) = 0$$

"Net" work is 0.

To compute $\int_C \bar{F}\cdot d\bar{r}$ it pays to ask whether $\bar{F}$ is conservative, and if it is, to find f.

Test whether $\bar{F} = \nabla f$. $\quad \vec{F} = \langle P, Q\rangle$. If it's conservative,

$$\bar{F} = \langle f_x, f_y\rangle = \langle P, Q\rangle.$$

$$P_y = f_{xy} = f_{yx} = Q_x$$

If given $\langle P, Q\rangle$, $P_y = Q_x$, then $\langle P, Q\rangle$ is conservative.

How to find f?

$$F = \langle \underset{P}{3+2xy}, \underset{Q}{x^2-3y^2} \rangle \qquad P_y = 2x \qquad Q_x = 2x$$

$$f_x = 3+2xy, \quad f = 3x + x^2y + \underbrace{g(y)}_{\text{"constant"}}$$

$$f_y = x^2-3y^2, \quad f = x^2y - y^3 + \underbrace{h(x)}$$

same?

$$g(y) = -y^3 \qquad h(x) = 3x$$

So $f = 3x + x^2y - y^3$.

If $\bar{F} = \langle P, Q, R \rangle$, test to see whether $\bar{F}$ is conservative.

$$\langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$$

$$P_y = f_{xy} = f_{yx} = Q_x \qquad P_z = f_{xz} = f_{zx} = R_x$$

$$Q_z = f_{yz} = f_{zy} = R_y$$

$$\bar{F} = \langle \underset{P}{y+z}, \underset{Q}{x+z}, \underset{R}{x+y} \rangle \qquad P_y = 1, \ Q_x = 1 \qquad P_z = 1, \ R_x = 1$$

$$Q_z = 1, \ R_y = 1$$

$$f_x = y+z, \quad f = xy + xz + \underbrace{g(y,z)}_{yz}$$

$$f_y = x+z, \quad f = xy + yz + \underbrace{h(xz)}_{xz}$$

$$f_z = x+y, \quad f = xz + yz + \underbrace{i(x,y)}_{xy}$$

$$f = xy + xz + yz$$

"$\bar{F}$ is conservative . . ."

Let $F = \langle \underset{P}{y+z}, \underset{Q}{x+z}, \underset{R}{x+y^2} \rangle$

$$\boxed{\begin{array}{l} P_y = 1, \ Q_x = 1 \\ Q_z = 1, \ R_y = 2y \end{array}} \quad \times$$

$$P_z = 1, \ R_x = 1$$

$f_x = y+z$, $f = xy + xz + \underset{y,z}{\underline{yz \qquad}} + y^2 z$

$f_y = x+z$, $f = xy + yz + \underset{x,z}{\underline{xz \qquad}} + \cancel{y^2 z}$

$f_z = x+y^2$, $f = xz + y^2 z + \underset{x,y}{\underline{xy \quad + \qquad}} \cancel{yz}$