To integrate a "derivative-like function" we can instead perform a calculation on the boundary of the region using original f.

$$\int_a^b f'\,dx = f(b) - f(a) \qquad \int_{\vec{a}}^{\vec{b}} \nabla f \cdot d\bar{r} = f(\bar{b}) - f(\bar{a})$$

<u>Green's Theorem</u> Suppose $F = \langle P, Q \rangle$ is a vector field, D is a region with boundary. Then

$$\iint_D Q_x - P_y\, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_C P\,dx + Q\,dy = \int_C \bar{F} \cdot d\bar{r}$$

if the curve C is traced out counterclockwise.

If $\bar{F}$ is conservative, $Q_x = P_y$. Then $\iint_D Q_x - P_y\, dA = \iint_D 0\, dA = 0$.

Also $\int_C \bar{F} \cdot d\bar{r} = 0$

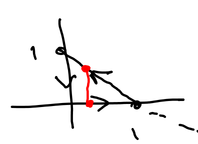
$C = \partial D =$ "boundary of D traced counterclockwise"

$$\int_{\partial D} \bar{F} \cdot d\bar{r} = \oint_C \bar{F} \cdot d\bar{r}$$

$$\int f'\,dx = f(b) - f(a)$$

$$\iint_D Q_x - P_y\, dA = \int_{\partial D} \bar{F} \cdot d\bar{r}$$

<u>Example</u> $\bar{F} = \langle \underset{P}{x^4}, \underset{Q}{xy} \rangle$, region D is




$y = -x + 1$

$P_y = 0 \quad Q_x = y$

$$\int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \frac{y^2}{2}\Big|_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2}\, dx = -\frac{(1-x)^3}{2 \cdot 3}\Big|_0^1 = 0 - -\frac{1}{6}$$
$$= \frac{1}{6}$$

$(0,0) \to (1,0)$ : $\underline{y = 0}$

$$\int P\,dx + \int Q\,dy = \int_0^1 x^4\,dx + \int_0^0 x\overset{0}{y}\,dy = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$$

$(1,0) \to (0,1)$ : $y = 1-x$ , $x = 1-y$

$$\int_1^0 x^4\,dx + \int_0^1 xy\,dy = \frac{x^5}{5}\Big|_1^0 + \int_0^1 (1-y)y\,dy = \left(0 - \frac{1}{5}\right) + \int_0^1 y - y^2\,dy$$

$$= -\frac{1}{5} + \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = -\frac{1}{5} + \frac{1}{2} - \frac{1}{3}$$

$(0,1) \to (0,0)$ , $x = 0$.

$$\int_0^0 \underset{0}{x^4}\,dx + \int_1^0 \underset{0}{x}y\,dy = 0$$

$$\int_{\partial D} \vec{F}\cdot d\vec{r} = \cancel{\frac{1}{5}} + -\cancel{\frac{1}{5}} + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$\iint_D 1\,dA$ = area of D. If $Q_x - P_y = 1$ then

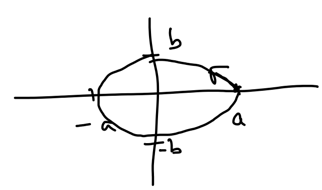
$$\iint_D 1\,dA = \int_{\partial D} \langle P, Q\rangle \cdot d\vec{r}$$

$P = 0,\ Q = x$ : $Q_x - P_y = 1 - 0 = 1$

$P = -y,\ Q = 0$ : $0 - -1 = 1$

$P = -\frac{y}{2},\ Q = \frac{x}{2}$ : $\frac{1}{2} - -\frac{1}{2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Area?



$$\iint_D 1\,dA = \int P\,dx + \int Q\,dy = \int -\frac{y}{2}\,dx + \int \frac{x}{2}\,dy$$

$$= \int \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle \cdot \bar{r}'\,dt$$

$$\bar{r}(t) = \langle \underline{a\cos(t)}, \underline{b\sin t} \rangle \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = 1$$

$$t : 0 \to 2\pi$$

$$\bar{r}' = \langle -a\sin(t), b\cos t \rangle \qquad \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle = \left\langle -\frac{b\sin t}{2}, \frac{a\cos t}{2} \right\rangle$$

$$\int_0^{2\pi} \left\langle -\frac{b\sin t}{2}, \frac{a\cos t}{2} \right\rangle \cdot \langle -a\sin t, b\cos t \rangle\,dt$$

$$= \int_0^{2\pi} \frac{ab\sin^2 t}{2} + \frac{ab\cos^2 t}{2}\,dt = \int_0^{2\pi} \frac{ab}{2}\,dt = \frac{ab}{2}t\Big|_0^{2\pi}$$

$$= \frac{ab}{2}2\pi = \pi ab.$$

If $a = b$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$. Area : $\pi a^2$