Divergence and curl, section 16.5

Gradient: $\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f \qquad \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Suppose $\vec{F}$ is a vector field.

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

This is the divergence of $\vec{F}$ at the point (x, y, z).

$$\nabla \times \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

This is the curl of $\vec{F}$ at (x, y, z).

$\vec{F} = \langle P, Q, R \rangle \qquad \nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

If $\nabla \times \vec{F} = \langle 0, 0, 0 \rangle$ then $\vec{F}$ is conservative.

<u>Facts</u>

$\nabla \cdot (\nabla \times \bar{F}) = 0: \quad \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

$\nabla \times (\nabla f) = \vec{0} \qquad \nabla \times (\nabla f) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} = \langle \quad , \quad , \quad \rangle$

always conservative

<u>Example</u> $\nabla \times \langle e^z, 1, xe^z \rangle \qquad \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & 1 & xe^z \end{vmatrix} = \langle 0-0, e^z - e^z, 0-0 \rangle$

$= \langle 0, 0, 0 \rangle$

So $\langle e^z, 1, xe^z \rangle$ is conservative.

Rewrite Green's Theorem:

1) $\vec{F}$ is a 2D vector field, but write $\vec{F} = \langle P, Q, 0 \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \langle 0-0, 0-0, Q_x - P_y \rangle$$

$$(\nabla \times \bar{F}) \cdot \underbrace{(0,0,1)}_{\bar{k}} = (\nabla \times \vec{F}) \cdot \bar{k} = Q_x - P_y .$$

$$\int_{\partial D} \bar{F} \cdot d\bar{r} = \iint_D (\nabla \times \vec{F}) \cdot \bar{k} \; dA$$

macroscopic swirl

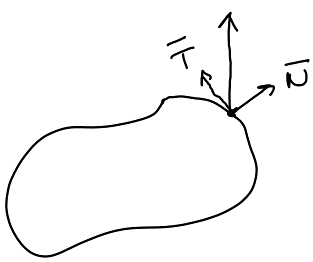
adding up "microscopic" swirl.

2) $\bar{r} = \langle x, y \rangle = \langle x(t), y(t) \rangle$

$\bar{r}' = \langle x', y' \rangle$

$\vec{T} = \dfrac{\langle x', y' \rangle}{|\bar{r}'|}$

$\vec{N} = \dfrac{\langle y', -x' \rangle}{|\bar{r}'|} \cdot \vec{T} = 0$



$$\int_{\partial D} \bar{F} \cdot \bar{N} \, ds = \int_{\partial D} \langle P, Q, 0 \rangle \cdot \frac{\langle y', -x', 0 \rangle}{\cancel{|\bar{r}'|}} \cancel{|\bar{r}'|} dt$$

$$= \int_{\partial D} (Py' - Qx') \, dt = \int_{\partial D} P \frac{dy}{dt} dt - Q \frac{dx}{dt} dt$$

$$= \int_{\partial D} P \, dy - Q \, dx = \int_{\partial D} -Q \, dx + P \, dy$$

$$= \iint_D P_x - -Q_y \, dA = \iint_D (P_x + Q_y) \, dA$$

$$= \iint_D \nabla \cdot \langle P, Q, 0 \rangle \, dA$$

$$\int_{\partial D} \bar{F} \cdot \bar{N} \, ds = \iint_D \nabla \cdot \vec{F} \, dA$$

macroscopic divergence

adding up the microscopic "divergence" — the tendency of the fluid to 'leave' or 'collect at' this point.