Surface integrals, section 16.7

If $\bar{r}(u,v)$ is a surface, area $= \iint_R \underbrace{|\bar{r}_u \times \bar{r}_v|\, du\, dv}_{\text{area of a tiny parallelogram}}$. — called dS.

ds = small length along a curve.

Area $= \iint_R 1\, dS$ — remember $\iint_R 1\, \underset{dx\,dy}{dA}$ = area of R.

Given $f(x,y,z)$: $\iint_R f(x,y,z)\, dS = \iint_R f(x(u,v), y(u,v), z(u,v))\, |\bar{r}_u \times \bar{r}_v|\, du\, dv$

$\bar{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Suppose a surface has density $\sigma(x,y,z)$. Total mass of surface:

$$M = \iint_R \underbrace{\underbrace{\sigma(x,y,z)}_{\text{density}}\ \underbrace{dS}_{\text{area}}}_{\text{tiny mass}} = \int_0^{2\pi}\int_0^{\pi} (2\cos u)^2\, a^2 \sin u\, du\, dv$$

Suppose sphere $x^2+y^2+z^2=4$ has $\sigma = z^2$.

$\bar{r}(u,v) = \langle 2\sin u\cos v, 2\sin u \sin v, 2\cos u \rangle$ = sphere.

$\bar{r}_u = \langle 2\cos u \cos v, 2\cos u \sin v, -2\sin u \rangle$

$\bar{r}_v = \langle -2\sin u \sin v, 2\sin u\cos v, 0 \rangle$

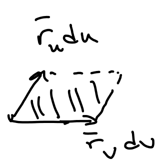
$\bar{r}_u \times \bar{r}_v = \langle 4\sin^2 u\cos v, 4\sin^2 u \sin v, 4\cos u \sin u \rangle$

$|\bar{r}_u \times \bar{r}_v| = a^2 \sin u$.

Suppose $\bar{F}$ is a vector field representing the velocity of a fluid in motion.

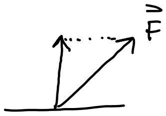
We have a surface. How fast is fluid flowing through the surface?

We look at a small patch of surface, bounded by $\bar{r}_u du$ & $\bar{r}_v dv$.



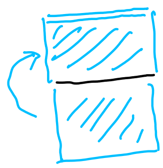
How fast is fluid moving in a $\perp$ direction across this patch?

Need: vector representing the component of velocity $\perp$ to the surface.

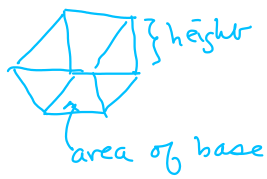


$\bar{r}_u \times \bar{r}_v$ is $\perp$ to surface.

Scalar projection: $\dfrac{\bar{F}\cdot\boxed{(\bar{r}_u \times \bar{r}_v)}}{\boxed{|\bar{r}_u \times \bar{r}_v|}} = \vec{F}\cdot\vec{N}$



$\bar{F}\cdot\bar{N}$

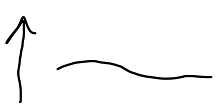


$\underbrace{\bar{F}\cdot\bar{N}}_{\text{height}}\ \underbrace{|\bar{r}_u \times \bar{r}_v|\,du\,dv}_{\text{area}}$

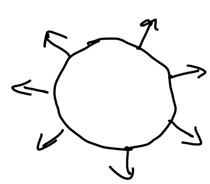
$$\text{Total} = \iint_R \bar{F}\cdot\underbrace{\bar{N}\,|\bar{r}_u \times \bar{r}_v|}\,du\,dv = \iint_R \bar{F}\cdot\langle \bar{r}_u \times \bar{r}_v\rangle\,du\,dv = \underline{\text{flux}}.$$

$$\underbrace{\bar{N}\,|\bar{r}_u \times \bar{r}_v|} = \frac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}\,|\bar{r}_u \times \bar{r}_v| = r_u \times r_v$$

We want positive flux to represent flux in one direction through the surface. We need $\bar{N}$ to point the same direction through the surface at every point.



To compute flux through the surface of a sphere — choose either <u>in</u> or <u>out</u>.



This all makes sense as long as the surface has 2 sides. Such a surface is an <u>orientable surface</u>.

Sphere of radius 2. $\vec{F} = \langle x, -y, z^2\rangle$, $r = \langle 2\sin u\cos v, 2\sin u\sin v, 2\cos u\rangle$

$\bar{r}_u \times \bar{r}_v = \langle 4\sin^2 u\cos v, 4\sin^2 u\sin v, 4\cos u\sin u\rangle$, $0 \le v \le 2\pi$, $0 \le u \le \pi$

Flux outward: z-coordinate $= 4\cos u\sin u$

Top hemisphere: $0 \le u < \frac{\pi}{2}$: $\cos u$ & $\sin u$ are positive, so z is pos.

Bottom hemisphere: $\frac{\pi}{2} < u \le \pi$: $\cos u < 0$, $\sin u > 0$, z is negative.

$$\text{Flux} = \int_0^{2\pi}\int_0^{\pi} \langle 2\sin u\cos v,\ -2\sin u\sin v,\ (2\cos u)^2\rangle \cdot \langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\cos u\sin u\rangle\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{\pi} \underbrace{8\sin^3 u\cos^2 v - 8\sin^3 u\sin^2 v}_{} + 4\cos^3 u\sin u\ du\, dv$$

$w = \cos u$

$dw = -\sin u\, du$

$$8\underbrace{\sin^3 u}_{}\,(\cos^2 v - \sin^2 v)$$

$\sin^2 u \sin u$

$(1-\cos^2 u)\sin u$

$w = \cos u$