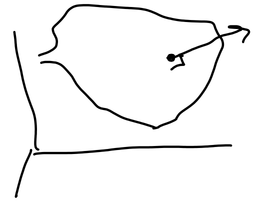
Stokes's Theorem, section 16.8

One form of Green's Theorem: $\displaystyle\int_{\partial D} \bar{F}\cdot d\bar{r} = \iint_D \underbrace{(\nabla\times\bar{F})\cdot\bar{k}}_{Q_x - P_y}\, dA$

$\bar{k} = \langle 0,0,1\rangle$ is $\perp$ region $D$.

Suppose $D$ is a surface with boundary $\partial D$.

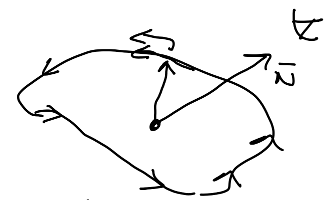


$\bar{N}$ a unit vector $\perp$ $D$.

In 3D: $\displaystyle\int_{\partial D} \bar{F}\cdot d\bar{r} = \iint_D (\nabla\times\bar{F})\cdot\bar{N}\; dS$

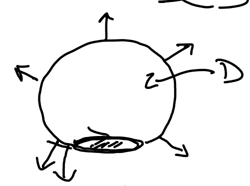
$\displaystyle\int_{\partial D} \bar{F}\cdot d\bar{r}$ represents flow around the boundary.

$\displaystyle\iint_D (\nabla\times\bar{F})\cdot\bar{N}\, dS$ "swirliness" at each point of $D$.

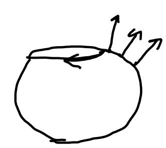



Stokes's Theorem: these are still equal.

Counterclockwise: take "up" to be the direction of $\bar{N}$.



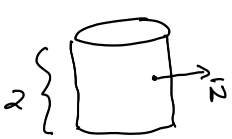

Imagine standing on the surface in the direction of $\bar{N}$. Walk to the boundary, turn left.



In Green's Theorem problems: usually use the double integral to calculate the single integral, still true unless we already know $\bar{F}$.

<u>Example</u> Cylinder $x^2+y^2=1$, $r = \langle \cos u, \sin u, v\rangle$, $0\le u\le 2\pi$, $0\le v\le 2$, oriented out.

$\bar{N}\,dS = \langle \bar{r}_u\times\bar{r}_v\rangle\, dv\,du$




$\bar{F} = \langle y, zx, xy\rangle$

$$\iint_D (\nabla\times F)\cdot\bar{N}\,dS = \int_0^{2\pi}\int_0^{2} \langle 0,-y,z-1\rangle\cdot\langle \cos u, \sin u, 0\rangle\, dv\,du$$

$$\begin{array}{ll} & \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \\ \times & \langle y, zx, xy\rangle \\ \hline \end{array}$$

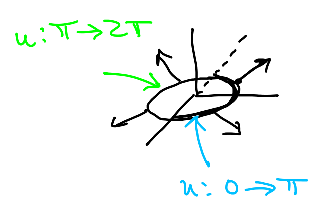
$= \langle x-x, 0-y, z-1\rangle$

$= \langle 0, -y, z-1\rangle$

$\bar{r}_u = \langle -\sin u, \cos u, 0\rangle$

$\bar{r}_v = \langle 0, 0, 1\rangle$

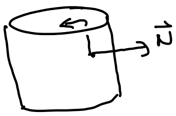
$\bar{r}_u\times\bar{r}_v = \langle \cos u, \sin u, 0\rangle$



$$= \int_0^{2\pi}\int_0^{2} -y\sin u \, dv\, du = \int_0^{2\pi}\int_0^{2} -\sin u \cdot \sin u \, dv\, du = \int_0^{2\pi}\int_0^{2} -\sin^2 u \, dv\, du$$

$$= -2\pi$$

$\Downarrow$



$\int_{\partial D} \bar{F}\cdot d\bar{r}$

$F = \langle y, zx, xy \rangle$

Top $\bar{r}(t) = \langle \cos t, \sin t, 2 \rangle$

$\bar{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$

$$\int_{2\pi}^{0} \langle \sin t, 2\cos t, \sin t \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_{2\pi}^{0} -\sin^2 t + 2\cos^2 t \, dt$$

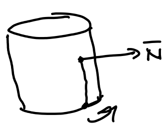
$$= \int_{2\pi}^{0} -(1-\cos^2 t) + 2\cos^2 t \, dt = \int_{2\pi}^{0} -1 + 3\cos^2 t \, dt$$

$$= \int_{2\pi}^{0} -1 + \frac{3}{2}(1+\cos 2t) \, dt = \int_{2\pi}^{0} \frac{1}{2} + \frac{3}{2}\cos(2t)\, dt$$

$$= \frac{1}{2}t + \frac{3}{4}\sin(2t) \Big|_{2\pi}^{0} = 0 - \left(\tfrac{1}{2}\pi + 0\right) = -\pi$$

Bottom $\bar{r} = \langle \cos t, \sin t, 0 \rangle$ $\quad F = \langle \sin t, 0, \cos t \sin t \rangle$

$\bar{r}' = \langle -\sin t, \cos t, 0 \rangle$



$$\int_0^{2\pi} \langle \sin t, 0, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t \, dt = \int_0^{2\pi} -\left(\frac{1-\cos(2t)}{2}\right) dt = \int_0^{2\pi} -\frac{1}{2} + \frac{\cos 2t}{2}\, dt$$

$$= -\frac{1}{2}t + \frac{\sin 2t}{4}\Big|_0^{2\pi} = -\frac{1}{2}2\pi = -\pi$$

Total: $-\pi - \pi = -2\pi$

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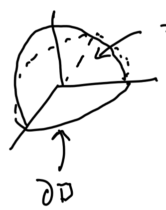
$$= \begin{cases} \int_{\partial D} \bar{F}\cdot d\bar{r} = \iint_D (\nabla\times\bar{F})\cdot \bar{N}\, dS \\ \int_{\partial E} \bar{F}\cdot d\bar{r} = \iint_E (\nabla\times\bar{F})\cdot \bar{N}\, dS \end{cases}$$

Possible: regions D & E have the same boundary: $\partial D = \partial E$.

If we want $\int_C \bar{F}\cdot d\bar{r}$, C is a closed curve that can be the boundary of a surface.

we can find <u>any</u> surface with C as the boundary.




D upper hemisphere. E = flat disk has the same boundary.

$\bar{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$ D

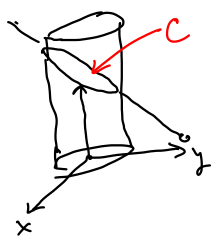
$\bar{r}(u,v) = \langle u\cos v, u \sin v, 0 \rangle$ E

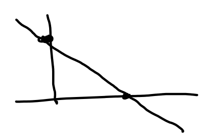
$1 - x^2 - y^2 = z$ G

$\bar{r}(u,v) = \langle \qquad \rangle$

<u>Example</u>

$\int_C \bar{F}\cdot d\bar{r}$, $F = \langle -y^2, x, z^2 \rangle$, $C$ = intersection of $x^2+y^2=1$ with $y+z=2. \Rightarrow z = 2-y$



$\bar{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$

$\int_0^{2\pi} \langle -y^2, x, z^2 \rangle \cdot \langle -\sin t, \cos t, -\cos t \rangle \, dt$

Surface = plane : $\bar{r}(u,v) = \langle v\cos u, v \sin u, 2 - v\sin u \rangle$

$\bar{r}_u \times \bar{r}_v = \langle 0, -v, -v \rangle$

$\nabla \times \bar{F} = \langle 0, 0, 1+2y \rangle$