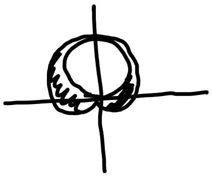
Assignment 19

15.2.7 Find the area inside $r = 1 + \sin\theta$ and outside $r = 2\sin\theta$.



$$\int_0^{2\pi}\int_0^{1+\sin\theta} r\,dr\,d\theta - \int_0^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{1+\sin\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{1+\sin\theta} d\theta = \int_0^{2\pi} \frac{(1+\sin\theta)^2}{2}\,d\theta = \frac{1}{2}\int_0^{2\pi} 1+2\sin\theta+\sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1+2\sin\theta+\frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{3}{2}+2\sin\theta-\frac{\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{1}{2}\left[\frac{3}{2}2\pi - 2(1) - 0 - 0 + 2(1) + 0\right]$$

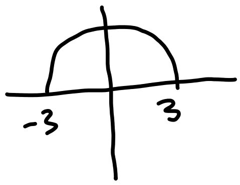
$$= 3\pi/2$$

$$\int_0^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{2\sin\theta} d\theta = \int_0^{2\pi} \frac{4\sin^2\theta}{2}\,d\theta = 2\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \int_0^{2\pi} 1-\cos 2\theta\,d\theta = \theta - \frac{1}{2}\sin 2\theta\Big|_0^{\pi} = \pi$$

Total: $3\pi/2 - \pi = \pi/2$

15.2.12 Compute $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx.$



$$\int_0^{\pi}\int_0^{3} \sin(r^2)\,r\,dr\,d\theta = \int_0^{\pi}\int_0^{9} \sin(u)\,\frac{du}{2}\,d\theta = \int_0^{\pi} -\frac{\cos u}{2}\Big|_0^9\,d\theta$$

$u = r^2$

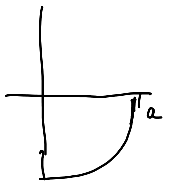
$du = 2r\,dr$

$\frac{du}{2} = r\,dr$

$$= \int_0^{\pi} -\frac{\cos 9}{2} + \frac{1}{2}\,d\theta = \frac{1-\cos 9}{2}\,\theta\Big|_0^{\pi}$$

$$= \frac{1-\cos 9}{2}\pi$$

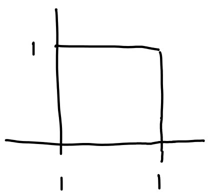
15.2.13 Compute $\displaystyle\int_0^a \int_{-\sqrt{a^2-x^2}}^{0} x^2 y\, dy\, dx.$



$$\int_{-\frac{\pi}{2}}^{0}\int_{0}^{a} r^2\cos^2\theta\, r\sin\theta\; r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{0} \frac{r^5}{5}\Big|_0^a \cos^2\theta\sin\theta\, d\theta$$

$$= \frac{a^5}{5}\int_{-\pi/2}^{0} \cos^2\theta\sin\theta\, d\theta = \frac{a^5}{5}\left[-\frac{\cos^3\theta}{3}\right]_{-\pi/2}^{0} = \frac{a^5}{5}\left[-\frac{1}{3}\right] = \frac{-a^5}{15}$$

15.3.1 Find the center of mass of a two-dimensional plate that occupies the square $[0,1] \times [0,1]$ and has density function xy.



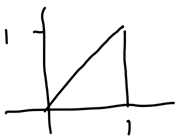
$$M = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 x \frac{y^2}{2} \Big|_0^1 dx = \frac{1}{2} \int_0^1 x \, dx = \frac{1}{2} \frac{x^2}{2} \Big|_0^1 = \frac{1}{4}$$

$$M_y = \int_0^1 \int_0^1 x^2 y \, dy \, dx = \int_0^1 x^2 \frac{1}{2} dx = \frac{1}{2} \frac{x^3}{3} \Big|_0^1 = \frac{1}{6}$$

$$M_x = \int_0^1 \int_0^1 x y^2 \, dy \, dx = \int_0^1 x \frac{y^3}{3} \Big|_0^1 dx = \int_0^1 x \frac{1}{3} dx = \frac{x^2}{2} \cdot \frac{1}{3} \Big|_0^1 = \frac{1}{6}$$

$$\bar{x} = \frac{M_y}{M} = \frac{1/6}{1/4} = \frac{2}{3} = \bar{y} \qquad \left(\frac{2}{3}, \frac{2}{3}\right) : \text{center of mass}$$

15.3.2 Find the center of mass of a two-dimensional plate that occupies the triangle $0 \le x \le 1$, $0 \le y \le x$, and has density function xy.



$$M = \int_0^1 \int_0^x xy \, dy \, dx = \int_0^1 x \frac{y^2}{2} \Big|_0^x dx = \int_0^1 x \frac{x^2}{2} dx = \int_0^1 \frac{x^3}{2} dx$$

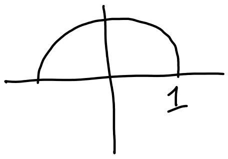
$$= \frac{x^4}{8} \Big|_0^1 = \frac{1}{8}.$$

$$M_y = \int_0^1 \int_0^x x^2 y \, dy \, dx = \int_0^1 x^2 \cdot \frac{x^2}{2} dx = \int_0^1 \frac{x^4}{2} dx = \frac{x^5}{10} \Big|_0^1 = \frac{1}{10}$$

$$M_x = \int_0^1 \int_0^x xy^2 \, dy \, dx = \int_0^1 x \frac{y^3}{3} \Big|_0^x dx = \int_0^1 \frac{x^4}{3} dx = \frac{x^5}{15} \Big|_0^1 = \frac{1}{15}$$

$$\bar{x} = \frac{1/10}{1/8} = \frac{8}{10} = \frac{4}{5} \qquad \bar{y} = \frac{1/15}{1/8} = \frac{8}{15}$$

15.3.3 Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function y.



$$M = \iint y\,dy\,dx = \int_0^{\pi}\int_0^{1} r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi} \frac{r^3}{3}\Big|_0^1 \sin\theta\,d\theta = \int_0^{\pi} \frac{1}{3}\sin\theta\,d\theta$$

$$= \frac{1}{3}(-\cos\theta)\Big|_0^{\pi} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

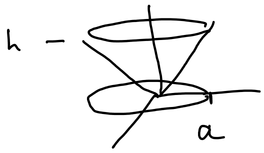
$$M_y = 0 \quad = \int_0^{\pi}\int_0^{1} r^2\cos\theta\sin\theta\, r\,dr\,d\theta$$

$$M_x = \int_0^{\pi}\int_0^{1} r^2\sin^2\theta\, r\,dr\,d\theta = \int_0^{\pi} \frac{r^4}{4}\Big|_0^1 \sin^2\theta\,d\theta = \frac{1}{4}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{8}\int_0^{\pi} 1-\cos 2\theta\,d\theta = \frac{1}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{1}{8}\left[\pi\right] = \frac{\pi}{8}$$

$$\bar{x} = \frac{0}{2/3} = 0\,,\quad \bar{y} = \frac{\pi/8}{2/3} = \frac{\pi}{8}\cdot\frac{3}{2} = \frac{3\pi}{16} \qquad \text{C of M: } \left(0, \frac{3\pi}{16}\right)$$

15.4.1 Find the area of the surface of a right circular cone of height h and base radius a.



$$f(x,y) = \frac{h}{a} r = \frac{h}{a}(x^2+y^2)^{1/2}$$

$$f_x = \frac{1}{2}\frac{h}{a}(x^2+y^2)^{-1/2}\, 2x = \frac{xh}{a\sqrt{x^2+y^2}}$$

$$f_y = \frac{yh}{a\sqrt{x^2+y^2}}$$

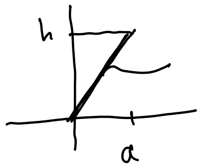
$$\sqrt{f_x^2+f_y^2+1} = \left(\frac{x^2h^2}{a^2(x^2+y^2)} + \frac{y^2h^2}{a^2(x^2+y^2)} + 1\right)^{1/2} = \left(\frac{h^2\cancel{(x^2+y^2)}}{a^2\cancel{(x^2+y^2)}} + 1\right)^{1/2}$$

$$= \left(\frac{h^2}{a^2}+1\right)^{1/2}$$

$$\iint \left(\frac{h^2}{a^2}+1\right)^{1/2} dy\,dx = \int_0^{2\pi}\int_0^a \left(\frac{h^2}{a^2}+1\right)^{1/2} r\,dr\,d\theta = \int_0^{2\pi} \left(\frac{h^2}{a^2}+1\right)^{1/2} \frac{r^2}{2}\Big|_0^a d\theta$$

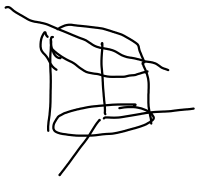
$$= \frac{a^2}{2}\left(\frac{h^2}{a^2}+1\right)^{1/2} \int_0^{2\pi} d\theta = \frac{a^2}{2}\left(\frac{h^2}{a^2}+1\right)^{1/2} \theta\Big|_0^{2\pi}$$

$$= 2\pi\frac{a^2}{2}\left(\frac{h^2}{a^2}+1\right)^{1/2} = \pi a^2\left(\frac{h^2+a^2}{a^2}\right)^{1/2} = \pi a\sqrt{a^2+h^2}$$



$\sqrt{a^2+h^2}$ = slant height of the cone.

15.4.2 Find the area of the portion of the plane $z = mx$ inside the cylinder $x^2 + y^2 = a^2$.



$f(x,y) = mx \quad f_x = m \quad f_y = 0$

$$\int_0^{2\pi}\int_0^{a} \sqrt{m^2+1}\, r\,dr\,d\theta = \sqrt{m^2+1}\int_0^{2\pi} \frac{r^2}{2}\Big|_0^{a} d\theta = \sqrt{m^2+1}\int_0^{2\pi} \frac{a^2}{2}\, d\theta$$

$$= \sqrt{m^2+1}\,\frac{a^2}{2}\,\theta\Big|_0^{2\pi} = 2\pi\frac{a^2}{2}\sqrt{m^2+1}$$

$$= \underline{\pi a^2\sqrt{m^2+1}}$$