Assignment 20

15.5.1 Evaluate $\int_0^1 \int_0^x \int_0^{x+y} 2x + y - 1 \, dz \, dy \, dx$.

$$\int_0^1 \int_0^x (2x+y-1)z \Big|_0^{x+y} dy\,dx = \int_0^1 \int_0^x (2x+y-1)(x+y)\, dy\, dx$$

$$= \int_0^1 \int_0^x 2x^2 + \underbrace{2xy + yx} + y^2 - x - y \, dy\,dx = \int_0^1 \int_0^x 2x^2 + 3xy + y^2 - x - y \, dy\,dx$$

$$= \int_0^1 2x^2 y + \frac{3}{2}xy^2 + \frac{y^3}{3} - xy - \frac{y^2}{2} \Big|_0^x dx = \int_0^1 2x^3 + \frac{3}{2}x^3 + \frac{x^3}{3} - x^2 - \frac{x^2}{2}\, dx$$

$$= \int_0^1 \left(2 + \frac{3}{2} + \frac{1}{3}\right) x^3 - \frac{3}{2}x^2 \, dx = \left(2 + \frac{3}{2} + \frac{1}{3}\right)\frac{x^4}{4} - \frac{3}{2}\frac{x^3}{3} \Big|_0^1$$

$$= \left(2 + \frac{3}{2} + \frac{1}{3}\right)\frac{1}{4} - \frac{1}{2} = \frac{11}{24}$$

15.5.2 Evaluate $\displaystyle\int_0^2 \int_{-1}^{x^2} \int_1^y xyz\,dz\,dy\,dx.$

$$\int_0^2\int_{-1}^{x^2} xy\,\frac{z^2}{2}\Big|_1^y\,dy\,dx = \int_0^2\int_{-1}^{x^2} \frac{xy^3}{2} - \frac{xy}{2}\,dy\,dx = \frac{1}{2}\int_0^2\int_{-1}^{x^2} xy^3 - xy\,dy\,dx =$$

$$\frac{1}{2}\int_0^2 \frac{xy^4}{4} - \frac{xy^2}{2}\Big|_{-1}^{x^2} dx = \frac{1}{2}\int_0^2 \frac{x^9}{4} - \frac{x^5}{2} - \frac{x}{4} + \frac{1}{2}x\,dx$$

$$= \frac{1}{2}\int_0^2 \frac{x^9}{4} - \frac{x^5}{2} + \frac{x}{4}\,dx = \frac{1}{2}\left[\frac{x^{10}}{40} - \frac{x^6}{12} + \frac{x^2}{8}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{2^{10}}{40} - \frac{2^6}{12} + \frac{2^2}{8}\right] = \frac{623}{60}.$$

15.5.3 Evaluate $\int_0^1 \int_0^x \int_0^{\ln y} e^{x+y+z}\, dz\, dy\, dx$. $= \int_0^1 \int_0^x \int_0^{\ln y} e^x e^y e^z \, dz\, dy\, dx$

$$= \int_0^1 \int_0^x e^x e^y e^z \Big|_0^{\ln y} dy\, dx = \int_0^1 \int_0^x e^x e^y e^{\ln y} - e^x e^y \, dy\, dx$$

$$= \int_0^1 \int_0^x e^x e^y y - e^x e^y \, dy\, dx = \int_0^1 e^x (y e^y - e^y) - e^x e^y \Big|_0^x dx$$

$$\int y e^y dy = uv - \int v\, du = y e^y - \int e^y dy = y e^y - e^y$$

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

$$= \int_0^1 e^x (x e^x - e^x) - e^x e^x - (e^x(0e^0 - e^0) - e^x e^0)\, dx$$

$$= \int_0^1 x e^{2x} - e^{2x} - e^{2x} - (-e^x - e^x)\, dx$$

$$= \int_0^1 x e^{2x} - 2e^{2x} + 2e^x dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} - e^{2x} + 2e^x \Big|_0^1$$

$$= \frac{1}{2} e^2 - \frac{1}{4} e^2 - e^2 + 2e - \left(-\frac{1}{4} e^0 - e^0 + 2e^0\right)$$

$$\int x e^{2x} dx = \frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x}$$

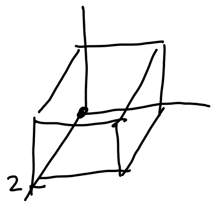
$u = x \quad dv = e^{2x} dx$

$du = dx \quad v = \frac{1}{2} e^{2x}$

$$= \left(\frac{1}{2} - \frac{1}{4} - 1\right) e^2 + 2e + \frac{1}{4} + 1 - 2$$

$$= -\frac{3}{4} e^2 + 2e - \frac{3}{4}$$

15.5.11 Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.



density $= x^2+y^2+z^2$

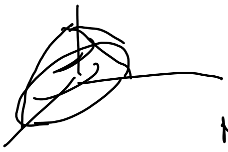
$$\text{Mass} = \int_0^2\int_0^2\int_0^2 x^2+y^2+z^2 \, dz\,dy\,dx = \int_0^2\int_0^2 x^2z+y^2z+\frac{z^3}{3}\Big|_0^2 dy\,dx$$

$$= \int_0^2\int_0^2 2x^2+2y^2+\frac{8}{3}\,dy\,dx = \int_0^2 2x^2y+\frac{2}{3}y^3+\frac{8}{3}y\Big|_0^2 dx$$

$$= \int_0^2 4x^2+\frac{16}{3}+\frac{16}{3}\,dx = \int_0^2 4x^2+\frac{32}{3}\,dx$$

$$= \frac{4}{3}x^3+\frac{32}{3}x\Big|_0^2 = \frac{4}{3}\cdot 8+\frac{32}{3}\cdot 2 = \frac{32}{3}+\frac{64}{3} = \frac{96}{3} = 32.$$

15.5.13 An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass.



density z :

$$M = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z \, dz\,dy\,dx$$

$$\bar{x} = \bar{y} = 0 \qquad \bar{z} = \frac{M_{xy}}{M} = \frac{\frac{64}{15}\pi}{4\pi} = \frac{16}{15}.$$

$$M_{xy} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2 \, dz\,dy\,dx$$

$$M = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{z^2}{2}\Big|_0^{\sqrt{4-x^2-y^2}} dy\,dx = \frac{1}{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4-x^2-y^2 \, dy\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (4-r^2)\, r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 4r - r^3 \, dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{4r^2}{2} - \frac{r^4}{4}\Big|_0^2 d\theta = \frac{1}{2}\int_0^{2\pi} \frac{4\cdot 4}{2} - \frac{2^4}{4}\, d\theta = \frac{1}{2}\int_0^{2\pi} 8 - 4 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4 \, d\theta = 2\theta\Big|_0^{2\pi} = 2\cdot 2\pi - 0 = 4\pi.$$

$$M_{xy} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2 \, dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{z^3}{3}\Big|_0^{\sqrt{4-x^2-y^2}} dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{3}(4-x^2-y^2)^{3/2} \, dy\,dx = \int_0^{2\pi}\int_0^2 \frac{1}{3}(4-r^2)^{3/2}\, r\,dr\,d\theta$$

$u = 4 - r^2$

$du = -2r\,dr$

$\frac{du}{-2} = r\,dr$

$$= \int_0^{2\pi}\int_4^0 \frac{1}{3} u^{3/2} \frac{du}{-2}\, d\theta = -\frac{1}{6}\int_0^{2\pi}\int_4^0 u^{3/2}\,du\,d\theta$$

$$= -\frac{1}{6}\int_0^{2\pi} \frac{2}{5} u^{5/2}\Big|_4^0 d\theta = -\frac{1}{6}\int_0^{2\pi} 0 - \frac{2}{5} 4^{5/2}\, d\theta$$

$$= -\frac{1}{6}\int_0^{2\pi} -\frac{2}{5} 32 \, d\theta = \frac{2}{6\cdot 5}\cdot 32\,\theta\Big|_0^{2\pi} = \frac{2\cdot 32\cdot 2\pi}{6\cdot 5} = \frac{64}{15}\pi$$