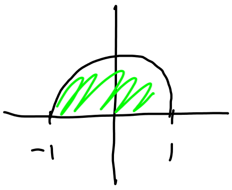
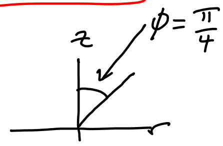
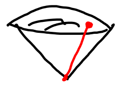
Assignment 21

15.6.2 Evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \boxed{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx.$



$z=\sqrt{x^2+y^2}=r$: cone





$z=\sqrt{2-x^2-y^2} \longrightarrow z^2=2-x^2-y^2 \longrightarrow x^2+y^2+z^2=2$

sphere of radius $\sqrt{2}$

Cone : $\phi=\frac{\pi}{4}$

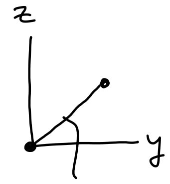
$$\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho \qquad \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$\int_0^{\pi}\int_0^{\pi/4} \frac{\rho^4}{4}\Big|_0^{\sqrt{2}} \sin\phi\, d\phi\, d\theta = \int_0^{\pi}\int_0^{\frac{\pi}{4}} \sin\phi\, d\phi\, d\theta = \int_0^{\pi} -\cos\phi\Big|_0^{\pi/4} d\theta$$

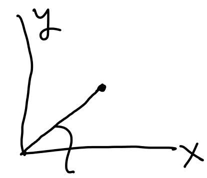
$$= \int_0^{\pi}\left(-\cos\tfrac{\pi}{4}+1\right)d\theta = \left(1-\cos\tfrac{\pi}{4}\right)\int_0^{\pi} d\theta = \left(1-\cos\tfrac{\pi}{4}\right)\theta\Big|_0^{\pi} = \left(1-\cos\tfrac{\pi}{4}\right)\pi$$

$$= \left(1-\frac{\sqrt{2}}{2}\right)\pi$$

15.6.12 An object occupies the region inside the unit sphere at the origin, and has density equal to the distance from the x-axis. Find the mass.



$\sqrt{y^2+z^2}$



$\sqrt{x^2+y^2}$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{y^2+z^2}\ \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{x^2+y^2}\ \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2}\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$\rho^2\sin^2\phi\cos^2\theta \qquad \rho^2\sin^2\phi\sin^2\theta)$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{(\rho^2\sin^2\phi)(\cos^2\theta+\sin^2\theta)}\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

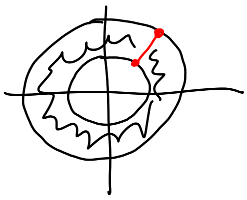
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho\sin\phi\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^3\sin^2\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^4}{4}\Big|_0^1 \sin^2\phi\, d\phi\, d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^{\pi} \frac{1-\cos 2\phi}{2}\, d\phi\, d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \phi - \frac{\sin 2\phi}{2}\Big|_0^{\pi} d\theta = \frac{1}{8}\int_0^{2\pi} (\pi - 0 - 0 + 0)\, d\theta$$

$$= \frac{\pi}{8}\,\theta\Big|_0^{2\pi} = \frac{\pi}{8}2\pi = \frac{\pi^2}{4}.$$

15.6.14 An object occupies the region between the unit sphere at the origin and a sphere of radius 2 with center at the origin, and has density equal to the distance from the origin. Find the mass.



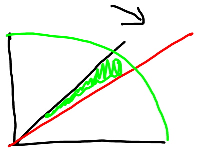
density $= \rho$

$$\int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho\ \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^4}{4}\Big|_1^2 \sin\phi \, d\phi \, d\theta$$

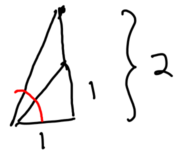
$$= \int_0^{2\pi}\int_0^{\pi}\left(\frac{16}{4}-\frac{1}{4}\right)\sin\phi \, d\phi \, d\theta = \frac{15}{4}\int_0^{2\pi} -\cos\phi \Big|_0^{\pi} d\theta$$

$$= \frac{15}{4}\int_0^{2\pi}(--1--1)\,d\theta = \frac{15}{2}\,\theta\Big|_0^{2\pi} = \frac{15}{2}\cdot 2\pi = 15\pi$$

15.6.15 An object occupies the region in the first octant bounded by the cones $\phi = \pi/4$ and $\phi = \arctan 2$, and the sphere $\rho = \sqrt{6}$, and has density proportional to the distance from the origin. Find the mass.



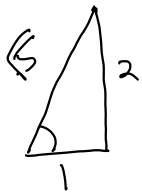
$$\int_0^{\pi/2}\int_{\pi/4}^{\arctan 2}\int_0^{\sqrt{6}} k\rho \quad \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



$$= \int_0^{\pi/2}\int_{\pi/4}^{\arctan 2} k \, \frac{\rho^4}{4}\Big|_0^{\sqrt{6}} \sin\phi \, d\phi \, d\theta = 9k \int_0^{\pi/2}\int_{\pi/4}^{\arctan 2} \sin\phi \, d\phi \, d\theta$$

$$= 9k \int_0^{\pi/2} -\cos\phi \Big|_{\pi/4}^{\arctan 2} d\theta = 9k \int_0^{\pi/2} \left(-\cos(\arctan 2) + \frac{\sqrt{2}}{2}\right) d\theta$$

$$= 9k\left(-\cos(\arctan 2) + \frac{\sqrt{2}}{2}\right) \theta \Big|_0^{\pi/2} = 9k\left(\frac{\sqrt{2}}{2} - \cos(\arctan 2)\right)\frac{\pi}{2}$$



cosine is $\frac{1}{\sqrt{5}}$

$$= \frac{9k\pi}{2}\left(\frac{\sqrt{2}}{2} - \frac{1}{\sqrt{5}}\right)$$

$\frac{\sqrt{5}}{5}$