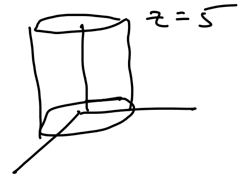
Assignment 22

Ex 15.6.4 Evaluate $\iiint xy\,dV$ over the interior of the cylinder $x^2+y^2=1$ between $z=0$ and $z=5$.



$$\int_0^{2\pi}\int_0^1\int_0^5 r\cos\theta\, r\sin\theta\, r\,dz\,dr\,d\theta$$

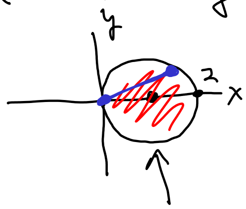
$$= \int_0^{2\pi}\int_0^1 r^3\cos\theta\sin\theta\; z\Big|_0^5 dr\,d\theta = 5\int_0^{2\pi}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$=5\int_0^{2\pi}\frac{r^4}{4}\Big|_0^1\cos\theta\sin\theta\,d\theta = \frac{5}{4}\int_0^{2\pi}\cos\theta\sin\theta\,d\theta$$

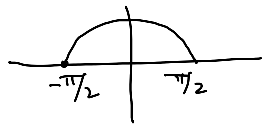
$$=\frac{5}{4}\;\frac{\sin^2\theta}{2}\Big|_0^{2\pi} = 0.$$

Ex 15.6.5 Evaluate $\iiint z\,dV$ over the region above the x-y plane, inside $x^2+y^2-2x=0$ and under $x^2+y^2+z^2=4$. $\Leftarrow$ $r^2+z^2=4,\ z=\sqrt{4-r^2}$

$$x^2-2x+1+y^2=0+1$$
$$(x-1)^2+y^2=1$$



$$r=2\cos\theta$$
$$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$



$$(r\cos\theta)^2+(r\sin\theta)^2-2r\cos\theta=0$$
$$r^2(\cos^2\theta+\sin^2\theta)-2r\cos\theta=0$$
$$r^2=2r\cos\theta$$
$$r=2\cos\theta$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\,\frac{z^2}{2}\Big|_0^{\sqrt{4-r^2}}dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\frac{r}{2}(4-r^2)\,dr\,d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}4r-r^3\,dr\,d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}2r^2-\frac{r^4}{4}\Big|_0^{2\cos\theta}d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}2\cdot4\cos^2\theta-\frac{1}{4}2^4\cos^4\theta\,d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}8\cos^2\theta-4\cos^4\theta\,d\theta$$

$$=2\int_{-\pi/2}^{\pi/2}2\cos^2\theta-\cos^4\theta\,d\theta=2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,(2-\cos^2\theta)\,d\theta$$

$$2-\cos^2\theta = 1+\underbrace{1-\cos^2\theta}_{\sin^2\theta}$$

$$=2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,(1+\sin^2\theta)\,d\theta=2\int_{-\pi/2}^{\pi/2}\cos^2\theta+\cos^2\theta\sin^2\theta\,d\theta$$

$$=2\int_{-\pi/2}^{\pi/2}\frac{1+\cos2\theta}{2}+\frac{1+\cos2\theta}{2}\cdot\frac{1-\cos2\theta}{2}\,d\theta$$

$$=\frac{2}{2}\int_{-\pi/2}^{\pi/2}1+\cos2\theta+\frac{1}{2}(1-\cos^2 2\theta)\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\frac{3}{2}+\cos2\theta-\frac{1}{2}\cdot\frac{1+\cos4\theta}{2}\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{3}{2} - \frac{1}{4} + \cos 2\theta - \frac{\cos 4\theta}{4} \, d\theta \qquad \frac{3}{2} - \frac{1}{4} = \frac{6}{4} - \frac{1}{4} = \frac{5}{4}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{5}{4} + \cos 2\theta - \frac{\cos 4\theta}{4} \, d\theta$$

$$= \left. \frac{5}{4}\theta + \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{16} \right|_{-\pi/2}^{\pi/2}$$

$$= \frac{5}{4}\frac{\pi}{2} + 0 - 0 + \frac{5}{4}\left(\frac{\pi}{2}\right) = \frac{5}{4}\frac{\pi}{2} + \frac{5}{4}\frac{\pi}{2} = \frac{5}{4}\pi$$

Ex 15.6.8 Evaluate $\iiint \sqrt{x^2+y^2}\,dV$ over the interior of $x^2+y^2+z^2=4$.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \left((\underline{\rho\sin\phi}\cos\theta)^2+(\underline{\rho\sin\phi}\sin\theta)^2\right)^{1/2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho\sin\phi\,(\cos^2\theta+\sin^2\theta)^{1/2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

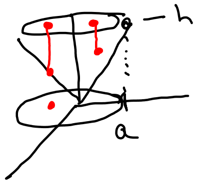
$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^3\sin^2\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^4}{4}\Big|_0^2 \sin^2\phi\, d\phi\, d\theta$$

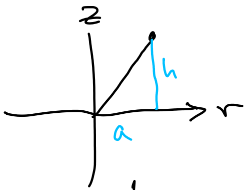
$$=\int_0^{2\pi}\int_0^{\pi} 4\sin^2\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} 4\,\frac{1-\cos 2\phi}{2}\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi} 2(1-\cos 2\phi)\, d\phi\, d\theta = \int_0^{2\pi} 2\left(\phi-\frac{\sin 2\phi}{2}\right)\Big|_0^{\pi} d\theta$$

$$=\int_0^{2\pi} 2(\pi)\, d\theta = 2\pi\,\theta\Big|_0^{2\pi} = 2\pi\cdot 2\pi = 4\pi^2$$

Ex 15.6.10 Find the mass of a right circular cone of height h and base radius a if the density is proportional to the distance from the base.





$z = \frac{h}{a} r$

density $= k(h - z)$

$$\int_0^{2\pi}\int_0^{a}\int_{\frac{h}{a}r}^{h} k(h-z)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a}\int_{\frac{h}{a}r}^{h} krh - krz \; dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a} krhz - kr\frac{z^2}{2} \Big|_{\frac{h}{a}r}^{h} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a} krh^2 - \frac{krh^2}{2} - krh\frac{h}{a}r + \frac{kr}{2}\frac{h^2}{a^2}r^2 \; dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a} \frac{krh^2}{2} - \frac{kh^2}{a}r^2 + \frac{kh^2}{2a^2}r^3 \; dr\, d\theta$$

$$= \int_0^{2\pi} \frac{kh^2}{2}\frac{r^2}{2} - \frac{kh^2}{a}\frac{r^3}{3} + \frac{kh^2}{2a^2}\frac{r^4}{4} \Big|_0^{a} d\theta$$

$$= \int_0^{2\pi} \frac{kh^2a^2}{4} - \frac{kh^2}{a}\frac{a^3}{3} + \frac{kh^2a^4}{8a^2} \; d\theta$$

$$= \int_0^{2\pi} kh^2a^2\left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) d\theta = \int_0^{2\pi} kh^2a^2\frac{1}{24}\, d\theta$$

$$= \frac{kh^2a^2}{24}\,\theta\Big|_0^{2\pi} = \frac{kh^2a^2}{24}2\pi = \frac{kh^2a^2\pi}{12}$$

Ex 15.6.11 Find the mass of a right circular cone of height h and base radius a if the density is proportional to the distance from its axis of symmetry.

density $= kr$

$$\int_0^{2\pi}\int_0^{a}\int_{\frac{h}{a}r}^{h} kr\; r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{a} kr^2 z\Big|_{\frac{h}{a}r}^{h} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a} kr^2\left(h-\frac{h}{a}r\right)dr\,d\theta = \int_0^{2\pi}\int_0^{a} khr^2 - \frac{kh}{a}r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} kh\frac{r^3}{3} - \frac{kh}{a}\frac{r^4}{4}\Big|_0^{a} d\theta = \int_0^{2\pi} kh\frac{a^3}{3} - \frac{kh}{a}\frac{a^4}{4}\,d\theta$$

$$= \int_0^{2\pi} kh\,a^3\left(\frac{1}{3}-\frac{1}{4}\right)d\theta = \frac{kha^3}{12}\,\theta\Big|_0^{2\pi} = \frac{kha^3}{12}2\pi = \frac{k\,ha^3}{6}\pi$$