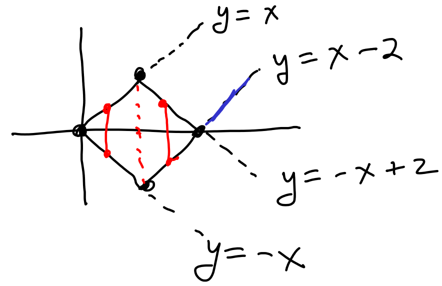
Assignment 23

Ex 15.7.2 Evaluate $\iint xy\,dx\,dy$ over the square with corners $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$ in two ways: directly, and using $x=(u+v)/2$, $y=(u-v)/2$. (answer)



$$x = \frac{u}{2} + \frac{v}{2} \qquad y = \frac{u}{2} - \frac{v}{2}$$

$f \qquad g$

$$f_u = \frac{1}{2} \qquad g_u = \frac{1}{2}$$

$$f_v = \frac{1}{2} \qquad g_v = -\frac{1}{2}$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

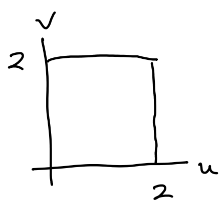
$y=x:\quad \cancel{\frac{u}{2}} - \frac{v}{2} = \cancel{\frac{u}{2}} + \frac{v}{2}, \quad 0 = v$

$y=-x:\quad \frac{u}{2} - \cancel{\frac{v}{2}} = -\frac{u}{2} - \cancel{\frac{v}{2}}, \quad u = 0$

$y=x-2:\quad \cancel{\frac{u}{2}} - \frac{v}{2} = \cancel{\frac{u}{2}} + \frac{v}{2} - 2, \quad 0 = v - 2$

$2 = v$

$y=-x+2:\quad \frac{u}{2} - \cancel{\frac{v}{2}} = -\frac{u}{2} - \cancel{\frac{v}{2}} + 2, \quad u = 2$



$$\int_0^2 \int_0^2 \left(\frac{u+v}{2}\right)\left(\frac{u-v}{2}\right) \frac{1}{2}\,dv\,du = \frac{1}{8} \int_0^2 \int_0^2 u^2 - v^2\,dv\,du$$

$$= \frac{1}{8} \int_0^2 u^2 v - \frac{v^3}{3} \Big|_0^2 du = \frac{1}{8} \int_0^2 2u^2 - \frac{8}{3}\,du$$

$$= \frac{1}{4} \int_0^2 u^2 - \frac{4}{3}\,du = \frac{1}{4}\left(\frac{u^3}{3} - \frac{4}{3}u\right)\Big|_0^2$$

$$= \frac{1}{4}\left(\frac{8}{3} - \frac{8}{3}\right) = 0$$

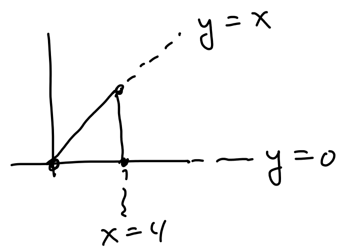
$$\int_0^1 \int_{-x}^{x} xy\,dy\,dx + \int_1^2 \int_{x-2}^{2-x} xy\,dy\,dx = 0 + 0 = 0$$

$$\int_0^1 \int_{-x}^{x} xy\,dy\,dx = \int_0^1 x\frac{y^2}{2}\Big|_{-x}^{x} dx = \int_0^1 \frac{x^3}{2} - x\frac{(-x)^2}{2}\,dx = \int_0^1 0\,dx = 0$$

$$\int_1^2 \int_{x-2}^{2-x} xy\,dy\,dx = \int_1^2 x\frac{y^2}{2}\Big|_{x-2}^{2-x} dx = \int_1^2 x\frac{(2-x)^2}{2} - x\frac{(x-2)^2}{2}\,dx$$

$$= \int_1^2 0\,dx = 0$$

Ex 15.7.6 Evaluate $\iint \sqrt{x^2+y^2}\,dx\,dy$ over the triangle with corners $(0,0)$, $(4,4)$, and $(4,0)$ using $x=u$, $y=uv$. (answer)



$x = u$

$y = uv$

$f_u = 1 \quad g_u = v$

$f_v = 0 \quad g_v = u$

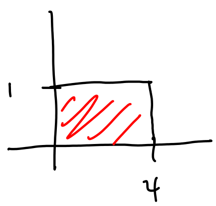
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = u$$

$y = 0:\ uv = 0,\quad \underline{\underline{u=0}}$ or $v = 0$

$x = 4:\ u = 4$

$y = x:\ uv = u,\quad uv - u = 0,\quad u(v-1) = 0$

$\underline{\underline{u = 0}}$ or $v = 1$



$$\int_0^4 \int_0^1 u\sqrt{1+v^2}\,u\,dv\,du$$

$$\sqrt{x^2+y^2} = \sqrt{u^2+u^2v^2} = u\sqrt{1+v^2}$$

$$= \int_0^4 u^2 \int_0^1 \sqrt{1+v^2}\,dv\,du$$

$$\int_0^1 \sqrt{1+v^2}\,dv = \int_0^{\pi/4} \underbrace{\sqrt{1+\tan^2(w)}}_{\sec^2(w)}\,\sec^2 w\,dw = \int_0^{\pi/4} \sec^3 w\,dw$$

$v = \tan(w)$

$dv = \sec^2(w)\,dw$

$v = 0,\ w = 0$

$v = 1,\ w = \frac{\pi}{4}$

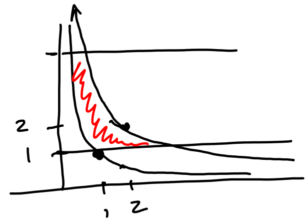
$$= \frac{\sec w \tan w}{2} + \frac{\ln|\sec w + \tan(w)|}{2}\Bigg|_0^{\pi/4}$$

$$= \frac{\sqrt{2}\cdot 1}{2} + \frac{1}{2}\ln|\sqrt{2}+1| - 0 - \frac{1}{2}\ln|1+0|$$

$$\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\ln|\sqrt{2}+1|\right)\int_0^4 u^2\,du = \left(\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right)\frac{u^3}{3}\Bigg|_0^4$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right)\frac{64}{3}$$

Ex 15.7.7 Evaluate $\iint y\sin(xy)\,dx\,dy$ over the region bounded by $xy=1$, $xy=4$, $y=1$, and $y=4$ using $x=u/v$, $y=v$. (answer)



$x=\frac{u}{v}$ $\quad y=v$

$f_u=\frac{1}{v}$ $\quad g_u=0$

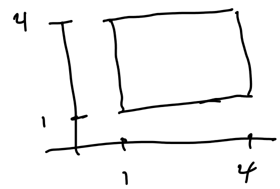
$f_v=\frac{-u}{v^2}$ $\quad g_v=1$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{v}$

$y=1:\ v=1$ $\qquad y=4:\ v=4$

$xy=1:\ \frac{u}{v}\cdot v=1\ ,\ u=1$

$xy=4:\ \frac{u}{v}\cdot v=4\ ,\ u=4$



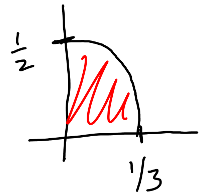
$$\int_1^4\int_1^4 \cancel{v}\sin(u)\ \frac{1}{\cancel{v}}\,dv\,du=\int_1^4 \sin(u)\,v\Big|_1^4\,du$$

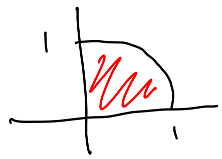
$$=3\int_1^4 \sin(u)\,du = 3(-\cos u)\Big|_1^4 = 3\big(-\cos(4)+\cos(1)\big)$$

Ex 15.7.8 Evaluate $\iint \sin(9x^2+4y^2)\,dA$, over the region in the first quadrant bounded by the ellipse $9x^2+4y^2=1$. (answer)

$x = \frac{u}{3} \qquad y = \frac{v}{2}$

$9x^2+4y^2 = 9\frac{u^2}{9} + 4\frac{v^2}{4} = u^2+v^2 = 1$





$f_u = \frac{1}{3} \quad g_u = 0$

$f_v = 0 \quad g_v = \frac{1}{2}$

$$\int_0^1 \int_0^{\sqrt{1-u^2}} \sin(u^2+v^2)\,\frac{1}{6}\,dv\,du$$

$u = r\cos\theta$
$v = r\sin\theta$
$u^2+v^2 = r^2$

$$\int_0^{\pi/2}\int_0^1 \sin(r^2)\,\frac{1}{6}\,r\,dr\,d\theta$$

$w = r^2$
$dw = 2r\,dr$
$r=0 \quad w=0$
$r=1 \quad w=1$

$$= \frac{1}{6}\int_0^{\pi/2}\int_0^1 \sin w\,\frac{dw}{2}\,d\theta$$

$$= \frac{1}{6\cdot 2}\int_0^{\pi/2} -\cos(w)\Big|_0^1\,d\theta$$

$$= \frac{1}{12}\int_0^{\pi/2} -\cos(1)+\cos(0)\,d\theta$$

$$= \frac{1}{12}(1-\cos(1))\,\theta\Big|_0^{\pi/2}$$

$$= \frac{1}{12}(1-\cos(1))\frac{\pi}{2}$$