Assignment 24

Ex 16.2.1 Compute $\int_C xy^2\,ds$ along the line segment from $(1,2,0)$ to $(2,1,3)$. (answer)

$$\bar{r}(t) = \langle 1,2,0\rangle + t\langle 1,-1,3\rangle = \langle 1+t,\ 2-t,\ 3t\rangle$$

$$\bar{r}' = \langle 1,-1,3\rangle$$

$$|\bar{r}'| = \sqrt{1+1+9}$$

$$\int_0^1 (1+t)(2-t)^2\sqrt{11}\,dt$$

$$= \sqrt{11}\int_0^1 4-3t^2+t^3\,dt$$

$$= \sqrt{11}\left[4t-t^3+\frac{t^4}{4}\right]_0^1 = \sqrt{11}\left(4-1+\frac{1}{4}\right) = \frac{13}{4}\sqrt{11}$$

Ex 16.2.2 Compute $\int_C \sin x\, ds$ along the line segment from $(-1, 2, 1)$ to $(1, 2, 5)$. (answer)

$\vec{r}(t) = \langle -1, 2, 1 \rangle + t\langle 2, 0, 4 \rangle$ $\qquad |r'| = \sqrt{4+0+16} = \sqrt{20}$

$$\int_0^1 \sin(2t-1)\sqrt{20}\, dt = \frac{\sqrt{20}}{2}\int_{-1}^{1} \sin(u)\, du = \frac{\sqrt{20}}{2}\left(-\cos u\right)\Big|_{-1}^{1}$$

$u = 2t-1$

$du = 2dt$

$\frac{du}{2} = dt$

$t = 0:\ u = -1$

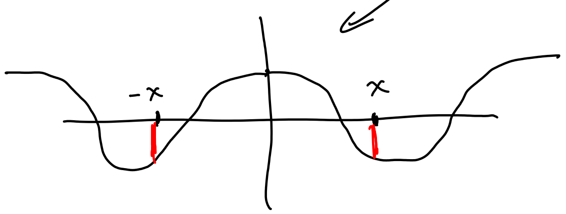
$t = 1:\ u = 2\cdot 1 - 1 = 1$

$$= \frac{\sqrt{20}}{2}\left(-\cos(1) - -\cos(-1)\right)$$

$$= \frac{\sqrt{20}}{2}\left(-\cos(1) + \cos(-1)\right)$$

$[\cos(x) = \cos(-x)]$

$$= \frac{\sqrt{20}}{2}\left(-\cos(1) + \cos(1)\right) = 0$$



Ex 16.2.4 Compute $\int_C \sin x\,dx + \cos y\,dy$ along the top half of the unit circle, from $(1,0)$ to $(-1,0)$. (answer)

$x: 1 \to -1$

$y: 0 \to 0$

$$\int_a^a f(x)\,dx = 0$$

$$\int_1^{-1} \sin x\,dx + \cancel{\int_0^0 \cos y\,dy}$$

$$= -\cos x \Big|_1^{-1} = -\cos(-1) - -\cos(1)$$

$$= -\cos(-1) + \cos(1)$$

$$= -\cos(1) + \cos(1) = 0$$

$$\int_C \sin x\,dx + \cos y\,dy = \int_C \langle \sin x, \cos y\rangle \cdot \bar{r}'\,dt$$

$$\bar{r} = \langle \overset{x}{\cos t}, \overset{y}{\sin t}\rangle$$

$t: 0 \to \pi$

$$\bar{r}' = \langle -\sin t, \cos t\rangle$$

$$\int_0^{\pi} \langle \sin(\cos t), \cos(\sin t)\rangle \cdot \langle -\sin t, \cos t\rangle\,dt$$

$$= \int_0^{\pi} -\sin t \sin(\cos t) + \cos t \cos(\sin t)\,dt$$

$$\int_0^{\pi} -\sin t \sin(\cos t)\,dt = \int_1^{-1} \sin u\,du = -\cos(u)\Big|_1^{-1}$$

$$= -\cos(-1) - -\cos(1) = 0$$

$u = \cos t$

$du = -\sin t\,dt$

$t = 0,\ u = 1$

$t = \pi,\ u = \cos(\pi) = -1$

$$\int_0^{\pi} \cos t \cos(\sin t)\,dt = \int_0^0 \cos u\,du = 0$$

$u = \sin t$

$du = \cos t\,dt$

$t = 0:\ u = 0$

$t = \pi:\ u = 0$

Ex 16.2.7 Compute $\int_C xe^y\,dx + x^2y\,dy$ along the curve $x = 3t,\ y = t^2$, $0 \le t \le 1$. (answer)

$r = \langle 3t, t^2 \rangle$ $y = t^2 = \left(\frac{3t}{3}\right)^2 = \left(\frac{x}{3}\right)^2 = \frac{x^2}{9}$

$t=0: (0,0)$

$t=1: (3,1)$ $x = 3t = 3\sqrt{t^2} = 3\sqrt{y}$

$$\int_0^3 xe^y\,dx + \int_0^1 x^2y\,dy$$

$$= \int_0^3 xe^{x^2/9}\,dx + \int_0^1 (3\sqrt{y})^2 y\,dy$$

$$\int_0^3 xe^{x^2/9}\,dx = \frac{9}{2}\int_0^1 e^u\,du = \frac{9}{2}\left[e^u\right]_0^1 = \frac{9}{2}(e-1)$$

$u = \frac{x^2}{9}$ $x=0,\ u=0$

$du = \frac{2}{9}x\,dx$ $x=3,\ u=1$

$\frac{9}{2}du = x\,dx$

$$\int_0^1 9y^2\,dy = 9\cdot\frac{y^3}{3}\Big|_0^1 = 9\cdot\frac{1}{3} = 3$$

$3 - \frac{9}{2} = \frac{6-9}{2} = \frac{-3}{2}$

Total $\frac{9}{2}e - \frac{9}{2} + 3 = \frac{9}{2}e - \frac{3}{2}$

Ex 16.2.11 Compute $\int_C \langle 1/xy, 1/(x+y)\rangle \cdot d\mathbf{r}$ along the curve $\langle 2t, 5t\rangle$, $1 \le t \le 4$. (answer)

$$r' = \langle 2, 5 \rangle$$

$$\int_1^4 \left\langle \frac{1}{xy}, \frac{1}{x+y} \right\rangle \cdot \langle 2, 5 \rangle \, dt = \int_1^4 \frac{2}{xy} + \frac{5}{x+y} \, dt$$

$$= \int_1^4 \frac{2}{10t^2} + \frac{5}{2t+5t} \, dt = \int_1^4 \frac{1}{5} t^{-2} + \frac{5}{7t} \, dt$$

$$= \frac{1}{5} \frac{t^{-1}}{-1} + \frac{5}{7} \ln(t) \Big|_1^4 = \frac{1}{5} \frac{4^{-1}}{-1} + \frac{5}{7} \ln(4) - \left[\frac{1}{5} \frac{1}{-1} + \cancelto{0}{\frac{5}{7} \ln(1)} \right]$$

$$= -\frac{1}{20} + \frac{5}{7} \ln(4) + \frac{1}{5} = \left(\frac{1}{5} - \frac{1}{20} \right) + \frac{5}{7} \ln(2^2)$$

$$= \left(\frac{4}{20} - \frac{1}{20} \right) + \frac{5}{7} \cdot 2 \ln(2)$$

$$= \frac{3}{20} + \frac{10}{7} \ln(2)$$

$$\ln(a^b) = b \ln(a)$$

Ex 16.2.15 An object moves from $(1, 1)$ to $(4, 8)$ along the path $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, subject to the force $\mathbf{F} = \langle x^2, \sin y \rangle$. Find the work done. (answer)

$r(t) = \langle t^2, t^3 \rangle$ $\quad (1,1)$ to $(4,8)$

$t=1$ to $t=2$

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_1^2 \langle x^2, \sin y \rangle \cdot \langle 2t, 3t^2 \rangle \, dt \qquad r' = \langle 2t, 3t^2 \rangle$$

$$= \int_1^2 2t\,(t^2)^2 + \sin(t^3)\, 3t^2 \, dt$$

$$\int_1^2 2t^5 \, dt = 2 \cdot \frac{t^6}{6} \Big|_1^2 = \frac{1}{3}\left[2^6 - 1\right] = \frac{63}{3} = 21$$

$$\int_1^2 \sin(t^3)\, 3t^2 dt = \int_1^8 \sin(u)\, du = -\cos u \Big|_1^8$$

$u = t^3$

$du = 3t^2 dt$

$t=1,\ u=1$

$t=2,\ u=8$

$$= -\cos(8) + \cos(1)$$

$$= \cos(1) - \cos(8)$$

Total work: $21 + \cos(1) - \cos(8)$

Ex 16.2.16 An object moves along the line segment from $(1,1)$ to $(2,5)$, subject to the force $\mathbf{F} = \langle x/(x^2+y^2), y/(x^2+y^2)\rangle$. Find the work done. (answer)

$$\bar{r}(t) = \langle 1,1\rangle + t\langle 1,4\rangle$$
$$= \langle 1+t, 1+4t\rangle$$
$$\bar{r}' = \langle 1,4\rangle$$

$$\int_0^1 \left\langle \frac{1+t}{(1+t)^2+(1+4t)^2}, \frac{1+4t}{(1+t)^2+(1+4t)^2}\right\rangle \cdot \langle 1,4\rangle\, dt$$

$$= \int_0^1 \frac{1+t+4(1+4t)}{(1+t)^2+(1+4t)^2}\,dt = \int_0^1 \frac{5+17t}{1+2t+t^2+1+8t+16t^2}\,dt$$

$$= \int_0^1 \frac{5+17t}{2+10t+17t^2}\,dt = \frac{1}{2}\int_2^{29} \frac{1}{u}\,du = \frac{1}{2}\ln u\Big|_2^{29}$$

$u = 2+10t+17t^2$

$du = 10+2\cdot 17t\ dt$

$= 2(5+17t)\ dt$

$\frac{du}{2} = 5+17t\,dt$

$t=0,\ u=2$

$t=1,\ u=2+10+17$

$=29$

$$= \frac{1}{2}(\ln 29 - \ln 2)$$