Assignment 29

Ex 16.6.4 Find the area of the portion of $2x + 4y + z = 0$ inside $x^2 + y^2 = 1$. (answer)

$$z = -2x - 4y$$

$$\bar{r} = \langle x, y, -2x - 4y \rangle$$

$$\bar{r}_x = \langle 1, 0, -2 \rangle$$

$$\bar{r}_y = \langle 0, 1, -4 \rangle$$

$$\bar{r}_x \times \bar{r}_y = \langle 2, 4, 1 \rangle$$

$$\iint \sqrt{4+16+1}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{21}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \sqrt{21}\, \frac{r^2}{2}\Big|_0^1 d\theta = \int_0^{2\pi} \sqrt{21}\,\frac{1}{2}\, d\theta = \frac{\sqrt{21}}{2}\,\theta\Big|_0^{2\pi}$$

$$= \frac{\sqrt{21}}{2}\, 2\pi = \sqrt{21}\,\pi$$

Ex 16.6.5 Find the area of $z = x^2 + y^2$ that lies below $z = 1$. (answer)

$$z = 1 : \quad 1 = x^2 + y^2$$

$$\bar{r} = \langle x, y, x^2 + y^2 \rangle$$

$$\bar{r}_x = \langle 1, 0, 2x \rangle$$

$$\bar{r}_y = \langle 0, 1, 2y \rangle$$

$$\bar{r}_x \times \bar{r}_y = \langle -2x, -2y, 1 \rangle$$

$$\iint \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{4r^2 + 1}\, r\,dr\,d\theta$$

$$u = 4r^2 + 1$$

$$du = 8r\,dr$$

$$\frac{du}{8} = r\,dr$$

$$r = 0,\ u = 1$$

$$r = 1,\ u = 5$$

$$= \int_0^{2\pi}\int_1^5 \sqrt{u}\, \frac{du}{8}\, d\theta$$

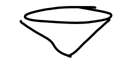
$$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3} u^{3/2}\Big|_1^5 d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3}\left(5^{3/2} - 1\right) d\theta = \frac{1}{4\cdot 3}\left(5^{3/2} - 1\right)\int_0^{2\pi} d\theta$$

$$= \frac{1}{4\cdot 3}\left(5^{3/2} - 1\right)\theta\Big|_0^{2\pi} = \frac{1}{4\cdot 3}\, 2\pi \left(5^{3/2} - 1\right)$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

Ex 16.6.6 Find the area of $z = \sqrt{x^2+y^2}$ that lies below $z = 2$.



$\bar r = \langle x, y, \sqrt{x^2+y^2} \rangle$

$\bar r_x = \left\langle 1, 0, \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \right\rangle$

$\bar r_y = \left\langle 0, 1, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$

$\bar r_x \times \bar r_y = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle$

$2 = \sqrt{x^2+y^2}$

$4 = x^2+y^2$

$$\iint \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\, dA$$

$$= \int_0^{2\pi}\int_0^{2} \sqrt{2}\quad r\,dr\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\, \frac{r^2}{2}\Big|_0^2 d\theta = \int_0^{2\pi} \sqrt{2}\, \frac{4}{2}\, d\theta$$

$$= 2\sqrt{2}\int_0^{2\pi} d\theta = 2\sqrt{2}\,\theta\Big|_0^{2\pi} = 2\sqrt{2}\,2\pi = 4\sqrt{2}\pi$$

Ex 16.6.7 Find the area of the portion of $x^2+y^2+z^2 = a^2$ that lies in the first octant. (answer)

$\rho = a$

$x = a\sin\phi\cos\theta$

$y = a\sin\phi\sin\theta$

$z = a\cos\phi$

u v u v u

$\bar r = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi \rangle$

$\bar r_\phi = \langle a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi \rangle$

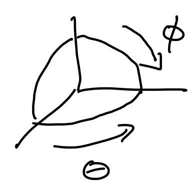
$\bar r_\theta = \langle -a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0 \rangle$

$\bar r_\phi \times \bar r_\theta = \langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\cos\phi\sin\phi\cos^2\theta + a^2\cos\phi\sin\phi\sin^2\theta \rangle$

$= \langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\cos\phi\sin\phi \rangle$

$|\bar r_\phi \times \bar r_\theta| = a^2\sin\phi\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi}$

$= a^2\sin\phi$



$$\int_0^{\pi/2}\int_0^{\pi/2} a^2\sin\phi\; d\phi\, d\theta = \int_0^{\pi/2} a^2(-\cos\phi)\Big|_0^{\pi/2} d\theta$$

$$= a^2\int_0^{\pi/2}\left(-\cos\frac{\pi}{2} + \cos(0)\right) d\theta = a^2\int_0^{\pi/2} d\theta = a^2\,\theta\Big|_0^{\pi/2} = a^2\frac{\pi}{2}$$

$$\int_0^{\pi/2}\int_0^{\pi/2} a^2\sin u\, du\, dv$$

Ex 16.6.9 Find the area of $z = x^2 - y^2$ that lies inside $x^2 + y^2 = a^2$.

$\bar{r} = \langle x, y, x^2 - y^2 \rangle$

$\bar{r}_x = \langle 1, 0, 2x \rangle$

$\bar{r}_y = \langle 0, 1, -2y \rangle$

$\bar{r}_x \times \bar{r}_y = \langle -2x, 2y, 1 \rangle$

$$\iint \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$\int_0^{2\pi}\int_0^{a} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$u = 4r^2 + 1$

$du = 8r\,dr$

$r = 0, u = 1$

$r = a, u = 4a^2 + 1$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^{4a^2+1} u^{1/2}\,du\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3} u^{3/2}\Big|_1^{4a^2+1} d\theta = \frac{1}{8}\frac{2}{3}\int_0^{2\pi} (4a^2+1)^{3/2} - 1\; d\theta$$

$$= \frac{1}{4\cdot 3}\left((4a^2+1)^{3/2} - 1\right)\theta\Big|_0^{2\pi} = 2\pi\frac{1}{4\cdot 3}\left((4a^2+1)^{3/2} - 1\right)$$

$$= \frac{\pi}{6}\left((4a^2+1)^{3/2} - 1\right)$$

$z = x^2 - y^2 \quad f_x = 2x \quad f_y = -2y$

$$\iint \sqrt{4x^2 + 4y^2 + 1}\; dy\,dx$$