Assignment 30

Ex 16.7.1 Find the center of mass of an object that occupies the upper hemisphere of $x^2 + y^2 + z^2 = 1$ and has density $x^2 + y^2$.

$\bar{x} = 0$

$\bar{y} = 0$

$\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$

$r_u \times r_v = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \rangle$

$|\bar{r}_u \times \bar{r}_v| = \sin u$

$$M = \iint (x^2+y^2)\,dS = \int_0^{2\pi}\int_0^{\pi/2} (\sin^2 u \cos^2 v + \sin^2 u \sin^2 v) \sin u \,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \sin^2 u \sin u \,du\,dv = \int_0^{2\pi}\int_0^{\pi/2} (1-\cos^2 u)\sin u\,du\,dv$$

$w = \cos u$ $\quad u: 0 \quad w: 1$

$dw = -\sin u\,du$ $\quad u: \frac{\pi}{2} \quad w: 0$

$$= \int_0^{2\pi}\int_1^0 -(1-w^2)\,dw\,dv = \int_0^{2\pi} w - \frac{w^3}{3}\Big|_0^1 dv = \int_0^{2\pi} 1 - \frac{1}{3}\,dv$$

$$= \frac{2}{3}\int_0^{2\pi} 1\,dv = \frac{2}{3} v \Big|_0^{2\pi} = \frac{2}{3}\cdot 2\pi = \frac{4}{3}\pi$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2} \cos u \sin^2 u \sin u \,du\,dv = \int_0^{2\pi}\int_0^{\pi/2} \sin^3 u \cos u\,du\,dv$$

$w = \sin u$

$dw = \cos u\,du$

$$= \int_0^{2\pi} \frac{\sin^4 u}{4}\Big|_0^{\pi/2} dv = \int_0^{2\pi} \frac{1}{4}\,dv = \frac{1}{4} v \Big|_0^{2\pi} = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi/2}{\frac{4}{3}\pi} = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$$

center of mass: $(0, 0, 3/8)$

Ex 16.7.6 Evaluate $\displaystyle\iint_D \langle x, y, 3\rangle \cdot \mathbf{N}\,dS$, where D is given by $z = 3x - 5y$, $1 \le x \le 2$, $0 \le y \le 2$, oriented up.

$\bar{N}\,dS = (r_u \times r_v)\,du\,dv$

$\bar{r} = \langle u, v, 3u - 5v \rangle$

$\bar{r}_u = \langle 1, 0, 3 \rangle$

$\bar{r}_v = \langle 0, 1, -5 \rangle$

$\bar{r}_u \times \bar{r}_v = \langle -3, 5, 1 \rangle$

$$\int_1^2 \int_0^2 \langle x, y, 3\rangle \cdot \langle -3, 5, 1\rangle \, dv\,du$$

$$= \int_1^2 \int_0^2 \langle u, v, 3\rangle \cdot \langle -3, 5, 1\rangle \, dv\,du$$

$$= \int_1^2 \int_0^2 -3u + 5v + 3 \, dv\,du$$

$$= \int_1^2 -3uv + \frac{5}{2}v^2 + 3v \Big|_0^2 du = \int_{+1}^2 -6u + 10 + 6 \, du$$

$$= -6\frac{u^2}{2} + 16u \Big|_{+1}^2 = -3 \cdot 4 + 32 - (-3 + 16)$$

$$= -12 + 32 + 3 - 16 = 16 + 3 - 12 = 7$$

Ex 16.7.7 Evaluate $\displaystyle\iint_D \langle x, y, -2\rangle \cdot \mathbf{N}\,dS$, where D is given by $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$, oriented up.

$\bar{r} = \langle x, y, 1 - x^2 - y^2 \rangle$

$\bar{r}_x = \langle 1, 0, -2x \rangle$

$\bar{r}_y = \langle 0, 1, -2y \rangle$

$\bar{r}_x \times \bar{r}_y = \langle 2x, 2y, 1 \rangle$

$$\iint_R \langle x, y, -2\rangle \cdot \langle 2x, 2y, 1\rangle \, dy\,dx$$

$$= \iint_R 2x^2 + 2y^2 - 2 \, dy\,dx$$

$\bar{r} = \langle u\cos v, u\sin v, 1 - u^2\cos^2 v - u^2\sin^2 v \rangle$

$= \langle u\cos v, u\sin v, 1 - u^2 \rangle$

$\bar{r}_u = \langle \cos v, \sin v, -2u \rangle$

$\bar{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$\bar{r}_u \times \bar{r}_v = \langle 2u^2\cos v, 2u^2\sin v, u\cos^2 v + u\sin^2 v \rangle$

$= \langle 2u^2\cos v, 2u^2\sin v, u \rangle$

$$\int_0^{2\pi} \int_0^1 \langle x, y, -2\rangle \cdot \langle 2u^2\cos v, 2u^2\sin v, u\rangle \, du\,dv$$

$$= \int_0^{2\pi}\int_0^1 (2r^2 - 2)\ r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^3 - 2r\ dr\,d\theta$$

$$= \int_0^{2\pi} \frac{2r^4}{4} - 2\frac{r^2}{2}\Big|_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2} - 1\right) d\theta = \left(-\frac{1}{2}\right)\theta\Big|_0^{2\pi}$$

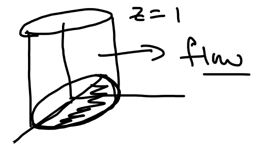
$$= -\frac{1}{2}2\pi = -\pi$$

$$= \int_0^{2\pi}\int_0^1 \langle u\cos v, u\sin v, -2\rangle \cdot \langle 2u^2\cos v, 2u^2\sin v, u\rangle\ du\,dv$$

$$= \int_0^{2\pi}\int_0^1 2u^3\cos^2 v + 2u^3\sin^2 v - 2u\ du\,dv$$

$$= \int_0^{2\pi}\int_0^1 2u^3 - 2u\ du\,dv$$

Ex 16.7.11 A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = \langle z, y^2, x^2 \rangle$, where distances are in meters and the components of $\mathbf{v}$ are in meters per second. Find the rate of flow outward through the portion of the cylinder $x^2 + y^2 = 4$, $0 \le z \le 1$ for which $y \ge 0$.



$\bar{v} = \bar{F} = \langle z, y^2, x^2 \rangle$ $\qquad \iint \langle z, y^2, x^2 \rangle \cdot (r_u \times r_v)\, du\, dv$

$\bar{r}(u,v) = \langle 2\cos v, 2\sin v, u \rangle \qquad 0 \le v \le \pi, \qquad 0 \le u \le 1$

$\bar{r}_u = \langle 0, 0, 1 \rangle$

$\bar{r}_v = \langle -2\sin v, 2\cos v, 0 \rangle$

$\bar{r}_u \times \bar{r}_v = \langle -2\cos v, -2\sin v, 0 \rangle \longrightarrow \langle 2\cos v, 2\sin v, 0 \rangle$

$$\int_0^\pi \int_0^1 \langle u, 4\sin^2 v, 4\cos^2 v \rangle \cdot \langle 2\cos v, 2\sin v, 0 \rangle\, du\, dv$$

$$= \int_0^\pi \int_0^1 2u\cos v + 8\sin^3 v\, du\, dv = \int_0^\pi u^2 \cos v + 8\sin^3 v\, u \Big|_0^1 dv$$

$$= \int_0^\pi \cos v + 8\sin^3 v\, dv = \int_0^\pi \cos v + 8(1-\cos^2 v)\sin v\, dv$$

$$= \int_0^\pi \cos v + 8\sin v - 8\cos^2 v \sin v\, dv = \sin v - 8\cos v + 8\frac{\cos^3 v}{3}\Big|_0^\pi$$

$w = \cos v$

$$= 0 - 8(-1) + 8\frac{-1}{3} - \left(0 - 8 + 8\frac{1}{3}\right)$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3} \text{ m}^3/\text{s}.$$

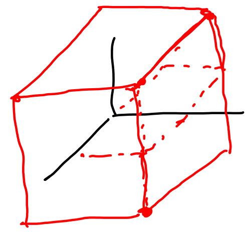
$$\frac{32}{3} \text{ m}^3/\text{s} \cdot 870 \text{ kg}/\text{m}^3 = \frac{32}{3} \cdot 870 \text{ kg/s}$$

$$= 9280 \text{ kg/s}.$$

Ex 16.7.12 Gauss's Law says that the net charge, Q, enclosed by a closed surface, S, is

$$Q = \epsilon_0 \iint \mathbf{E} \cdot \mathbf{N}\, dS$$

where $\mathbf{E}$ is an electric field and ϵ_0 (the permittivity of free space) is a known constant; $\mathbf{N}$ is oriented outward. Use Gauss's Law to find the charge contained in the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\mathbf{E} = \langle x, y, z \rangle$.



Top. $\bar{r} = \langle u, v, 1 \rangle \qquad -1 \le u \le 1,\ -1 \le v \le 1$

$\bar{r}_u = \langle 1, 0, 0 \rangle$

$\bar{r}_v = \langle 0, 1, 0 \rangle$

$\bar{r}_u \times \bar{r}_v = \langle 0, 0, 1 \rangle$ up ✓

$$\int_{-1}^{1}\int_{-1}^{1} \langle u, v, 1 \rangle \cdot \langle 0, 0, 1 \rangle \, du\, dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} 1 \, du\, dv = \int_{-1}^{1} u\Big|_{-1}^{1} dv = \int_{-1}^{1} 2\, dv$$

$$= 2v\Big|_{-1}^{1} = 4$$

Back $\bar{r} = \langle -1, u, v \rangle$

$\bar{r}_u = \langle 0, 1, 0 \rangle$

$\bar{r}_v = \langle 0, 0, 1 \rangle$

$r_u \times r_v = \langle 1, 0, 0 \rangle$ toward front,

so use $\langle -1, 0, 0 \rangle$

$$\int_{-1}^{1}\int_{-1}^{1} \langle -1, u, v \rangle \cdot \langle -1, 0, 0 \rangle \, du\, dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} 1 \, du\, dv = 4$$

$6 \cdot 4\, \varepsilon_0 = 24\, \varepsilon_0$