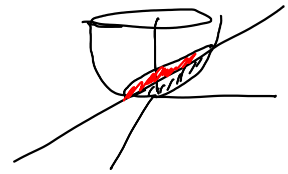
Assignment 31

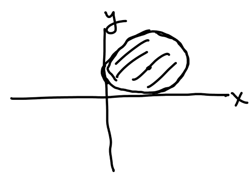
Ex 16.8.1 Let $\mathbf{F} = \langle z, x, y\rangle$. The plane $z = 2x + 2y - 1$ and the paraboloid $z = x^2 + y^2$ intersect in a closed curve. Stokes's Theorem implies that

$$\iint_{D_1} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{D_2} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS,$$

where the line integral is computed over the intersection C of the plane and the paraboloid, and the two surface integrals are computed over the portions of the two surfaces that have boundary C (provided, of course, that the orientations all match). Compute all three integrals. (answer)



$$2x+2y-1 = x^2+y^2$$
$$1+1-1 = x^2-2x+1+y^2-2y+1$$
$$1 = (x-1)^2+(y-1)^2$$



$$\oint_C \bar{F}\cdot d\bar{r} = \int_{\partial D} \bar{F}\cdot d\bar{r}$$

$$\bar{F} = \langle z, x, y\rangle$$

$2+2\cos t \quad 2+2\sin t - 1$

$$\bar{r}(t) = \langle 1+\cos t, 1+\sin t, 2(1+\cos t)+2(1+\sin t)-1\rangle$$
$$= \langle 1+\cos t, 1+\sin t, (1+\cos t)^2+(1+\sin t)^2\rangle$$
$$= \langle 1+\cos t, 1+\sin t, 1+2\cos t+\cos^2 t+1+2\sin t+\sin^2 t\rangle$$
$$= \langle 1+\cos t, 1+\sin t, 3+2\cos t+2\sin t\rangle$$
$$r' = \langle -\sin t, \cos t, -2\sin t+2\cos t\rangle$$

$$\int_0^{2\pi} \langle 3+2\cos t+2\sin t, 1+\cos t, 1+\sin t\rangle\cdot\langle -\sin t, \cos t, -2\sin t+2\cos t\rangle\, dt$$

$$= \int_0^{2\pi} -5\sin t+3\cos t+1-5\sin^2 t\, dt = \int_0^{2\pi} -5\sin t+3\cos t+1-\frac{5}{2}(1-\cos 2t)\, dt$$

$$= \int_0^{2\pi} -5\sin t+3\cos t-\frac{3}{2}+\frac{5}{2}\cos 2t\, dt$$

$$= +5\cos t+3\sin t-\frac{3}{2}t+\frac{5}{2}\frac{\sin 2t}{2}\Big]_0^{2\pi} = 5-\frac{3}{2}2\pi-(5) = -3\pi$$

Surface integral over the plane: $z = 2x+2y-1$

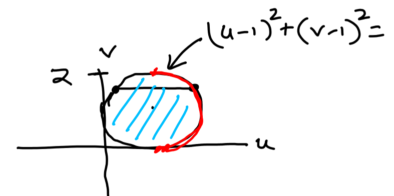
$$\iint (\nabla\times F)\cdot \underbrace{N\,dS}_{(r_u\times r_v)\,du\,dv}$$

$$\bar{r}(u,v) = \langle u, v, 2u+2v-1\rangle$$
$$\bar{r}_u = \langle 1, 0, 2\rangle$$
$$\bar{r}_v = \langle 0, 1, 2\rangle$$
$$\bar{r}_u\times\bar{r}_v = \langle -2, -2, 1\rangle$$

$$\begin{array}{l} \langle \partial/\partial x \quad \partial/\partial y \quad \partial/\partial z\rangle \\ \times \langle z \quad x \quad y\rangle \\ \hline \langle 1-0, 1-0, 1-0\rangle \end{array}$$

$$= \langle 1, 1, 1\rangle$$

$$\int_0^2 \int_{-\sqrt{1-(v-1)^2}+1}^{\sqrt{1-(v-1)^2}+1} \langle 1,1,1\rangle\cdot\langle -2,-2,1\rangle\, du\, dv$$



$$u = \pm\sqrt{1-(v-1)^2}+1$$

$$-3\int_0^2\int_{-\sqrt{1-(v-1)^2}+1}^{\sqrt{1-(v-1)^2}+1} 1\,du\,dv = -3\cdot \text{area of region} = -3(\pi 1^2) = -3\pi$$

$$= -3\int_0^2 u\Big|_{-\sqrt{1-(v-1)^2}+1}^{\sqrt{1-(v-1)^2}+1} dv = -3\int_0^2 \sqrt{1-(v-1)^2}+1-\left(-\sqrt{1-(v-1)^2}+1\right)dv$$

$$= -3\int_0^2 2\sqrt{1-(v-1)^2}\,dv$$

$v-1 = \sin w$

$dv = \cos w\,dw$

$v:0 \quad -1 = \sin w$

$v:2 \quad 1 = \sin w$

$$= -6\int_{-\pi/2}^{\pi/2} \cos w \cdot \cos w\,dw$$

$z = x^2+y^2$ $\quad \bar{r}(u,v) = \langle u, v, u^2+v^2\rangle$ $\quad \nabla\times F = \langle 1,1,1\rangle$

$\bar{r}_u = \langle 1, 0, 2u\rangle$

$\bar{r}_v = \langle 0, 1, 2v\rangle$

$\bar{r}_u \times \bar{r}_v = \langle -2u, -2v, 1\rangle$

$$\int_0^2\int_{-\sqrt{1-(u-1)^2}+1}^{\sqrt{1-(v-1)^2}+1} \langle 1,1,1\rangle\cdot\langle -2u,-2v,1\rangle\,dv\,du$$

$$\int_0^2\int -2u-2v+1\,dv\,du = \int_0^2 -2uv - 2\frac{v^2}{2} + v\Big|\,du$$

$$= \int_0^2 -2u\left(\sqrt{1-(u-1)^2}+1\right) - \underline{\left(\sqrt{1-(u-1)^2}+1\right)^2} + \left(\sqrt{1-(u-1)^2}+1\right) -$$
$$\left(-2u\left(-\sqrt{1-(u-1)^2}+1\right) - \underline{\left(-\sqrt{1-(u-1)^2}+1\right)^2} + \left(-\sqrt{1-(u-1)^2}+1\right)\right)du$$

$$= \int_0^2 -4u\sqrt{1-(u-1)^2} - 2\sqrt{1-(u-1)^2}\,du$$

$u-1 = \sin w$

$du = \cos w\,dw$

$$= \int_{-\pi/2}^{\pi/2} \left[-4(1+\sin w)\cos w - 2\cos w\right]\cos w\,dw$$

$$= \int_{-\pi/2}^{\pi/2} -4\cos^2 w - 4\sin w\cos^2 w - 2\cos^2 w\,dw$$

$$= \int_{-\pi/2}^{\pi/2} -6\cos^2 w - 4\sin w\cos^2 w\,dw = -3\pi$$

Ex 16.8.2 Let D be the portion of $z = 1 - x^2 - y^2$ above the x-y plane, oriented up, and let $\mathbf{F} = \langle xy^2, -x^2y, xyz\rangle$. Compute $\displaystyle\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$. (answer)

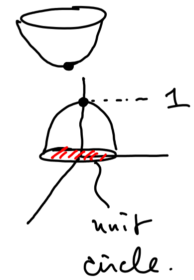
$z = x^2+y^2$

$z = -x^2-y^2$

$z = 1-x^2-y^2$

$0 = 1-x^2-y^2$

$x^2+y^2=1$



$\bar{r}(u,v) = \langle v\cos u, v\sin u, 0\rangle$

$\bar{r}_u = \langle -v\sin u, v\cos u, 0\rangle$

$\bar{r}_v = \langle \cos u, \sin u, 0\rangle$

$\bar{r}_u \times \bar{r}_v = \langle 0, 0, -v\sin^2 u - v\cos^2 u\rangle$

$= \langle 0, 0, -v\rangle$

use: $\langle 0,0,v\rangle$, oriented up.

$\langle \partial/\partial x \quad \partial/\partial y \quad \partial/\partial z\rangle$

$\times \langle xy^2 \quad -x^2y \quad xyz\rangle$

$\langle xz, -yz, -2xy-2xy\rangle$

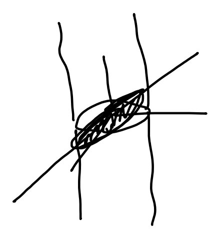
$= \langle xz, -yz, -4xy\rangle$

$$\int_0^{2\pi}\int_0^1 \langle 0, 0, -4v\cos u\, v\sin u\rangle \cdot \langle 0,0,v\rangle\, dv\, du$$

$$\int_0^{2\pi}\int_0^1 -4v^3 \cos u \sin u\, dv\, du = \int_0^{2\pi} -4\frac{v^4}{4}\Big|_0^1 \cos u \sin u\, du$$

$$= \int_0^{2\pi} (-1)\cos u \sin u\, du = (-1)\frac{\sin^2 u}{2}\Big|_0^{2\pi} = 0$$

Ex 16.8.3 Let D be the portion of $z = 2x + 5y$ inside $x^2 + y^2 = 1$, oriented up, and let $\mathbf{F} = \langle y, z, -x\rangle$. Compute $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$. (answer)



$\bar{r}(t) = \langle \cos t, \sin t, 2\cos t + 5\sin t\rangle$

$\bar{r}' = \langle -\sin t, \cos t, -2\sin t + 5\cos t\rangle$

$$\int_0^{2\pi} \langle \sin t, 2\cos t + 5\sin t, -\cos t\rangle \cdot \langle -\sin t, \cos t, -2\sin t + 5\cos t\rangle\, dt$$

$$\int_0^{2\pi} 7\sin t\cos t - 1 - 2\cos^2 t\, dt = \int_0^{2\pi} 7\sin t\cos t - 1 - \cancel{2}\frac{1+\cos 2t}{\cancel{2}}\, dt$$

$$= \int_0^{2\pi} 7\sin t\cos t - 2 - \cos 2t\, dt = \frac{7}{2}\sin^2 t - 2t - \frac{\sin 2t}{2}\Big|_0^{2\pi}$$

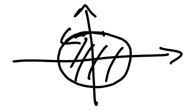
$$= 0 - 2\cdot 2\pi - 0 - (0 - 0 - 0)$$

$$= -4\pi$$

$\bar{r}(u,v) = \langle v\cos u, v\sin u, 2v\cos u + 5v\sin u\rangle$

Ex 16.8.4 Compute $\oint_C x^2 z\,dx + 3x\,dy - y^3\,dz$, where C is the unit circle $x^2+y^2=1$ oriented counter-clockwise. (answer)

$D =$ solid unit disk



$\bar{r} = \langle \cos t, \sin t, 0 \rangle$

$\bar{r}' = \langle -\sin t, \cos t, 0 \rangle$

$\int_0^{2\pi} \langle 0, 3\cos t, -\sin^3 t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$

$\int_0^{2\pi} 0 + 3\cos^2 t + 0 \, dt = \int_0^{2\pi} \frac{3}{2}(1+\cos 2t)\,dt$

$= \frac{3}{2}\int_0^{2\pi} 1 + \cos 2t \, dt = \frac{3}{2}\left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$

$= \frac{3}{2} 2\pi = 3\pi$

$\oint_C x^2 z\,dx + 3x\,dy - y^3\,dz$

$= \oint_C \langle x^2 z, 3x, -y^3 \rangle \cdot \langle dx, dy, dz \rangle$

$= \oint_C \langle x^2 z, 3x, -y^3 \rangle \cdot \underbrace{\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle}_{r'} dt$

$\bar{r}(u,v) = \langle v\cos u, v\sin u, 0 \rangle \ldots$

Ex 16.8.7 Show that $\int_C f\nabla g + g\nabla f \cdot d\mathbf{r} = 0$, where $\mathbf{r}$ describes a closed curve C to which Stokes's Theorem applies. (See theorems 12.4.1 and 16.5.2.)

$\longrightarrow$ $u \times (v+w) = u \times v + u \times w$

& $u \times (av) = a(u \times v)$

$\nabla \times \nabla f = 0$

$\int_{\partial D} (f\nabla g + g\nabla f) \cdot d\bar{r} = \iint_D \nabla \times (f\nabla g + g\nabla f) \cdot (r_u \times r_v)\,du\,dv$

$= \iint_D (\nabla \times f\nabla g + \nabla \times g\nabla f) \cdot (r_u \times r_v)\,du\,dv$

$= \int\int_D (f(\nabla \times \nabla g) + g(\nabla \times \nabla f)) \cdot (r_u \times r_v)\,du\,dv$

$= \iint_D \vec{0} \cdot \langle r_u \times r_v \rangle\,du\,dv = \iint_D 0\,du\,dv = 0$