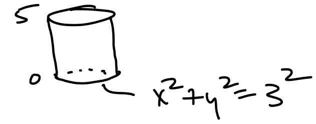
Assignment 32

Ex 16.9.1 Using $\mathbf{F} = \langle 3x, y^3, -2z^2 \rangle$ and the region bounded by $x^2+y^2=9$, $z=0$, and $z=5$, compute both integrals from the Divergence Theorem.

$$\iiint \nabla\cdot\bar{F}\,dV = \iiint 3 + 3y^2 + -4z\,dV$$

region



$$= \int_0^{2\pi}\int_0^3\int_0^5 \left(3+3(r\sin\theta)^2 - 4z\right) r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^5 3r + 3r^3\sin^2\theta - 4zr\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 3rz + 3r^3\sin^2\theta\, z - 4r\frac{z^2}{2}\Big|_0^5\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 15r + 15r^3\sin^2\theta - 2r\cdot 25\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{15}{2}r^2 + \frac{15}{4}r^4\sin^2\theta - 50\frac{r^2}{2}\Big|_0^3\,d\theta$$

$$= \int_0^{2\pi} \frac{15\cdot 9}{2} + \frac{15\cdot 81}{4}\sin^2\theta - 25\cdot 9\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{15\cdot 9}{2} - 25\cdot 9\right) + \frac{15\cdot 81}{4}\,\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{15\cdot 9}{2} - 25\cdot 9 + \frac{15\cdot 81}{8}\right) - \frac{15\cdot 81}{4}\cos 2\theta\,d\theta$$

$$= \left(\frac{15\cdot 9}{2} - 25\cdot 9 + \frac{15\cdot 81}{8}\right)\theta - \frac{15\cdot 81}{4}\,\frac{\sin 2\theta}{2}\Big|_0^{2\pi}$$

$$= \left(\frac{15\cdot 9}{2} - 25\cdot 9 + \frac{15\cdot 81}{8}\right)(2\pi) = -\frac{45}{4}\pi$$

Top + side + bottom : $\iint F\cdot N\,dS$ $\qquad F = \langle 3x, y^3, -2z^2\rangle$

Top:

$\bar{r} = \langle u\cos v, u\sin v, 5\rangle$

$\bar{r}_u = \langle \cos v, \sin v, 0\rangle$

$\bar{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

$\bar{r}_u \times \bar{r}_v = \langle 0, 0, u\cos^2 v + u\sin^2 v\rangle$

$= \langle 0, 0, u\rangle$

$$\int_0^{2\pi}\int_0^3 \langle 3u\cos v, u^3\sin^3 v, -50\rangle\cdot\langle 0,0,u\rangle\,du\,dv$$

$$= \int_0^{2\pi}\int_0^3 -50u\,du\,dv = -25\cdot 9\cdot 2\pi$$

Bottom $\bar{r} = \langle u\cos v, u\sin v, 0\rangle$

$\bar{r}_u \times \bar{r}_v = \langle 0, 0, u\rangle$

use $\langle 0, 0, -u\rangle$ oriented down

$$\int_0^{2\pi}\int_0^{3} \langle 3u\cos v, u^3\sin^3 v, 0\rangle \cdot \langle 0,0,-u\rangle \, du\, dv$$

$$= \int_0^{2\pi}\int_0^{3} 0 \, du\, dv = 0$$

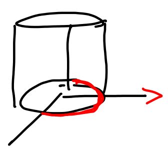
Side $\bar{r} = \langle 3\cos u, 3\sin u, v\rangle$

$$\int_0^{2\pi}\int_0^{5}$$

$\bar{r}_u = \langle -3\sin u, 3\cos u, 0\rangle$

$\bar{r}_v = \langle 0, 0, 1\rangle$

$\bar{r}_u \times \bar{r}_v = \langle 3\cos u, 3\sin u, 0\rangle$



$F = \langle 9\cos u, 27\sin^3 u, -4v^2\rangle$

$$\int_0^{2\pi}\int_0^{5} \langle 9\cos u, 27\sin^3 u, -4v^2\rangle \cdot \langle 3\cos u, 3\sin u, 0\rangle \, dv\, du$$

$$= \int_0^{2\pi}\int_0^{5} 27\cos^2 u + 81\sin^4 u \, dv\, du = \int_0^{2\pi} (27\cos^2 u + 81\sin^4 u)\, v\Big|_0^5 \, du$$

$$= \int_0^{2\pi} 5\cdot 27\cos^2 u + 5\cdot 81\sin^4 u \, du$$

$$= \int_0^{2\pi} 5\cdot 27\, \frac{1+\cos 2u}{2} + 5\cdot 81\left(\frac{1-\cos 2u}{2}\right)^2 du$$

$$= \int_0^{2\pi} \frac{5\cdot 27}{2} + \frac{5\cdot 27}{2}\cos 2u + \frac{5\cdot 81}{4}\left(1 - 2\cos 2u + \cos^2 2u\right) du$$

$$= \int_0^{2\pi} \frac{5\cdot 27}{2} + \frac{5\cdot 27}{2}\cos 2u + \frac{5\cdot 81}{4}\left(1 - 2\cos 2u + \frac{1+\cos 4u}{2}\right) du$$

$$= \int_0^{2\pi} \frac{5\cdot 27}{2} + \frac{5\cdot 27}{2}\cos 2u + \underbrace{\frac{5\cdot 81}{4}\left(\frac{3}{2}\right.}_{} - 2\cos 2u + \left.\frac{\cos 4u}{2}\right) du$$

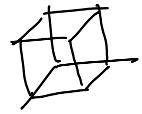
$$= \int_0^{2\pi} \frac{5\cdot 27}{2} + \frac{5\cdot 81\cdot 3}{8} + \left(\frac{5\cdot 27}{2} + \frac{5\cdot 81}{4}(-2)\right)\cos 2u + \frac{5\cdot 81}{8}\cos 4u \, du$$

$$= \left(\frac{5\cdot 27}{2} + \frac{5\cdot 81\cdot 3}{8}\right)u + \underbrace{\left(\frac{5\cdot 27}{2} - \frac{5\cdot 81}{2}\right)\frac{\sin 2u}{2} + \frac{5\cdot 81}{8}\,\frac{\sin 4u}{4}}_{}\Bigg|_0^{2\pi}$$

$$= \left(\frac{5\cdot 27}{2} + \frac{5\cdot 81\cdot 3}{8}\right)2\pi$$

Top + bottom + side $= -25\cdot 9\cdot 2\pi + 0 + \left(\frac{5\cdot 27}{2} + \frac{5\cdot 81\cdot 3}{8}\right)2\pi = -\frac{45}{4}\pi$

Ex 16.9.3 Let E be the volume described by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and $\mathbf{F} = \langle 2xy, 3xy, ze^{x+y} \rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$.



$$\iiint \nabla \cdot F\, dV = \int_0^1\int_0^1\int_0^1 2y + 3x + e^{x+y}\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 (2y+3x+e^{x+y})\, z\Big|_0^1 dy\, dx = \int_0^1\int_0^1 2y+3x+e^{x+y}\, dy\, dx$$

$e^x e^y$

$$= \int_0^1 y^2 + 3xy + e^x e^y \Big|_0^1 dx = \int_0^1 1 + 3x + e^x \cdot e - (e^x)\, dx$$

$$= x + \frac{3}{2}x^2 + e \cdot e^x - e^x \Big|_0^1 = 1 + \frac{3}{2} + e \cdot e - e - (e-1)$$

$$= \frac{5}{2} + e^2 - 2e + 1 = \frac{7}{2} + e^2 - 2e$$

Ex 16.9.5 Let E be the volume described by $x^2 + y^2 + z^2 \le 4$, and $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$.

$$\iiint \nabla \cdot F\, dV = \iiint_D 3x^2 + 3y^2 + 3z^2\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (3x^2+3y^2+3z^2)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3\rho^2 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} 3 \cdot \frac{\rho^5}{5} \sin\phi \Big|_0^2 d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{3 \cdot 2^5}{5} \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \frac{3\cdot 2^5}{5}(-\cos\phi)\Big|_0^{\pi} d\theta = \int_0^{2\pi} \frac{3 \cdot 2^5}{5}(1+1)\, d\theta$$

$$= \frac{3 \cdot 2^6}{5}\theta \Big|_0^{2\pi} = \frac{3\cdot 2^6}{5} 2\pi = \frac{3 \cdot 2^7}{5}\pi.$$

Ex 16.9.6 Let E be the hemisphere described by $0 \leq z \leq \sqrt{1-x^2-y^2}$, and $\mathbf{F} = \langle \sqrt{x^2+y^2+z^2}, \sqrt{x^2+y^2+z^2}, \sqrt{x^2+y^2+z^2} \rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$.

$$\nabla \cdot F = \frac{x}{\sqrt{x^2+y^2+z^2}} + \frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi)\rho\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (\sin\phi\cos\theta + \sin\phi\sin\theta + \cos\phi)\sin\phi \left.\frac{\rho^3}{3}\right|_0^1 d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (\sin\phi\cos\theta + \sin\phi\sin\theta + \cos\phi)\sin\phi \frac{1}{3}\, d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin^2\phi\cos\theta + \sin^2\phi\sin\theta + \sin\phi\cos\phi \, d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} (\cos\theta+\sin\theta)\left(\frac{1-\cos 2\phi}{2}\right) + \sin\phi\cos\phi \, d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \frac{\cos\theta+\sin\theta}{2} - \frac{(\cos\theta+\sin\theta)}{2}\cos 2\phi + \sin\phi\cos\phi \, d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \frac{\cos\theta+\sin\theta}{2}\phi - \frac{\cos\theta+\sin\theta}{2}\,\frac{\sin 2\phi}{2} + \left.\frac{\sin^2\phi}{2}\right|_0^{\pi/2} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \frac{\cos\theta+\sin\theta}{4}\pi + \frac{1}{2} - (0)\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \frac{\pi}{4}\cos\theta + \frac{\pi}{4}\sin\theta + \frac{1}{2}\, d\theta = \frac{1}{3}\left[-\frac{\pi}{4}\sin\theta - \frac{\pi}{4}\cos\theta + \frac{1}{2}\theta\right]_0^{2\pi}$$

$$= \frac{1}{3}\left[-\frac{\pi}{4} + \pi - \left(-\frac{\pi}{4}\right)\right] = \frac{1}{3}(+\pi) = \frac{+\pi}{3}$$