$f: A \to B \quad X \subseteq A \quad f(X) = \{f(x) \mid x \in X\}$

$Y \subseteq B$, $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$

$W, X \subseteq A$, $f(W \cap X) \subseteq f(W) \cap f(X)$

$b \in f(W \cap X)$, so there is $a \in W \cap X$,
$f(a) = b$. So $a \in W$, $a \in X$. $f(a) \in f(W)$,
$f(a) \in f(X)$, so $\underset{b}{\underset{\shortparallel}{f(a)}} \in f(W) \cap f(X)$

$f(1) = 1 \qquad W = \{1, 2\}$
$f(2) = 2 \qquad X = \{3, 4\}$
$f(3) = 2 \qquad f(W) = \{1, 2\} \quad f(X) = \{2, 3\}$
$f(4) = 3 \qquad f(W) \cap f(X) = \{2\}$

$f(W \cap X) \overset{?}{=} \emptyset \neq \{2\}$

$f: A \to B$

f is 1-1 or injective or an injection
if whenever $f(x) = f(y)$ then $x = y$.
or if $x \neq y$ then $f(x) \neq f(y)$.

f is onto or surjective or a surjection
if $\forall b \in B \, \exists a \in A \, (f(a) = b)$.
or $f(A) = B$ or range of f is B.

<u>THM</u> $f: A \to B$ $g: B \to C$: if f & g
are injective so is $g \circ f$.

Pf/ Suppose $(g \circ f)(a) = (g \circ f)(a')$.
$g(f(a)) = g(f(a'))$
Since g is 1-1, $f(a) = f(a')$.
Since f is 1-1, $a = a'$.

THM If f, g surjective, so is $g \circ f : A \to C$

Pf/ Pick $c \in C$. Want is $a \in A$ such that

$(g \circ f)(a) = c$.

Since g is onto, $\exists b \in B \; (g(b) = c)$.

Since f is onto, $\exists a \in A \; (f(a) = b)$.

$$g(f(a)) = g(b) = c$$

$$(g \circ f)(a) = c \quad \checkmark$$

THM If $g \circ f$ is 1-1 then f is 1-1.

If $g \circ f$ is onto then g is onto.

Pf/ Suppose $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, so $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f$ is 1-1, $x = y$.

$$(f \circ g) \circ h = f \circ (g \circ h)$$

THM If g is 1-1 $g \circ f_1 = g \circ f_2$ then $f_1 = f_2$

If f is onto & $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$.

Pf/ Pick some $b \in B$. Since f is onto, $\exists x \, (f(x) = b)$.

$$(g_1 \circ f)(x) = (g_2 \circ f)(x)$$

$$g_1(f(x)) = g_2(f(x))$$

$$g_1(b) = g_2(b)$$