At least 2 different infinite cardinal numbers: $\aleph_0$, c.

The collection of cardinal numbers is too big to be a set.

There are an infinite number of different infinite cardinal numbers.

$\mathcal{P}(A)$ = set of all subsets of A.

$A = \{1,2,3\}$, $\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$|A| = 3$, $|\mathcal{P}(A)| = 8 = 2^3$

If $A = \{a_1 \ldots a_n\}$. Either a_1 is in, or not; a_2 is in, or not; a_3 is in, or not; ... a_n is in, or not.

Total # ways to do this: $2 \cdot 2 \cdot 2 \cdots 2 = 2^n$.

Given A we want $\mathcal{P}(A)$ to be bigger.

THM $\overline{A} < \overline{\mathcal{P}(A)}$

Proof $\overline{A} \leq \overline{\mathcal{P}(A)}$ and $\overline{A} \neq \overline{\mathcal{P}(A)}$

Injection $f: A \to \mathcal{P}(A)$: $f(a) = \{a\}$.

We need to show there is no bijection from $A \to \mathcal{P}(A)$.

Suppose $g: A \to \mathcal{P}(A)$ is a bijection.

$S = \{a \in A \mid a \notin g(a)\}$

If, for example, $g(a) = \{a, b, c\}$ then $a \in g(a)$, so $a \notin S$.

or maybe $g(a) = \{b, d, w, x\}$, $a \notin g(a)$, $a \in S$.

$S \subseteq A$ so $S \in \mathcal{P}(A)$. Since g is a bijection, there is an element $x \in A$ such that $g(x) = S$.

1) If $x \in S$, $x \notin g(x) = S$ so $x \notin S$. contradiction.

2) If $x \notin S$ then $x \in g(x) = S$, so $x \in S$. contradiction.

So in fact, such a g does not exist.

$\overline{\mathbb{N}} < \overline{\mathcal{P}(\mathbb{N})} < \overline{\mathcal{P}(\mathcal{P}(\mathbb{N}))} < \cdots$

Imagine all sets. This is not a set.

Suppose that A = set of all sets.

Every element of $\mathcal{P}(A)$ is a set, so if $x \in \mathcal{P}(A)$, $x \in A$. So $\mathcal{P}(A) \subseteq A$. So $\overline{\overline{\mathcal{P}(A)}} \leq \overline{\overline{A}}$ $[f: \mathcal{P}(A) \to A$ by $f(x) = x]$

$\overline{\overline{A}} \leq \overline{\overline{\mathcal{P}(A)}}$. So $\overline{\overline{A}} = \overline{\overline{\mathcal{P}(A)}}$, contradiction.

<u>Set theory</u> Is there a set A such that $\overline{\overline{\mathbb{N}}} < \overline{\overline{A}} < \overline{\overline{\mathbb{R}}}$

In the 1920's, Kurt Gödel proved that the statement: "there is no such A", cannot be disproved.

In the 1960's, Paul Cohen proved that this statement cannot be proved true.

"there is no such A" is the continuum hypothesis.

Axioms of set theory + continuum hyp. OK

Axioms of set theory + $\neg$ continuum hyp OK.