Are there uncountable sets. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

The real numbers are not countable.

By contradiction — diagonalization method of Cantor.

Suppose $\mathbb{R}$ is countable: $\mathbb{R} = \{r_1, r_2, r_3, \cdots\}$

$$r_1 = \sim\sim . d_{11} d_{12} d_{13} d_{14} \cdots = \sim .3\,2\,1\,1\,6\,0\,0 \cdots$$
$$r_2 = \sim\sim . d_{21} d_{22} d_{23} d_{24} \cdots = \sim .4\,1\,2\,3\,7\,7\,7 \cdots$$
$$r_3 = \sim\sim . d_{31} d_{32} d_{33} \qquad = \sim .5\,3\,5\,1\,0\,0\,6 \cdots$$
$$r_4 = \sim\sim . d_{41} d_{42} d_{43} d_{44} \cdots = \sim .1\,1\,1\,3\,4\,2 \cdots$$
$$\vdots$$

Let $r = 0.d_1 d_2 d_3 \cdots$ $\qquad r = .1511\ldots$

$d_i = 1$ if $d_{ii} \neq 1$

$d_i = 5$ if $d_{ii} = 1$

$r$ does not appear (in this form) on the list, contradiction.

Some real numbers have more than 1 decimal expansion.

$$\underline{.0123\overline{0}} = .0122\overline{9} = \frac{1}{100} + \frac{2}{1000} + \frac{2}{10^4} + \underbrace{\frac{9}{10^5} + \frac{9}{10^6} + \cdots}$$
$$= \frac{1}{100} + \frac{2}{10^3} + \frac{2}{10^4} + \frac{9}{10^5}\underbrace{\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)}_{\text{geometric series}}$$

So $\mathbb{R}$ is not countable.

What does it mean that one infinite set is smaller than another.

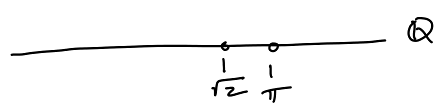
The size (cardinality) of $\{a, b, c\}$ is 3.

The cardinality of $\mathbb{N}$ is $\overline{\mathbb{N}}$. $\overline{\mathbb{N}} = \overline{\mathbb{Q}}$ because

$\mathbb{N} \approx \mathbb{Q}$.

$\overline{\mathbb{N}} = \overline{\mathbb{Q}} = \overline{\mathbb{Z}} = \overline{\{\text{even integers}\}} = \aleph_0$ = "aleph nought" = "aleph zero"

$\overline{\mathbb{R}} = c$ "continuum"



$\aleph_0 = \overline{\mathbb{N}} \le \overline{\mathbb{R}} = c$

Def. $\overline{A} \le \overline{B}$ if there is an injection $f: A \to B$.

$\overline{\mathbb{N}} \le \overline{\mathbb{Q}}$, $\overline{\mathbb{Q}} \le \overline{\mathbb{N}}$,

$\overline{\mathbb{N}} \le \overline{\mathbb{R}}$ : $f: \mathbb{N} \to \mathbb{R}$, $f(n) = n$

Suppose $\overline{A} = \overline{A'}$, $\overline{B} = \overline{B'}$, $\overline{A} \le \overline{B}$. Is it true that $\overline{A'} \le \overline{B'}$? Is there an injection $g: A' \to B'$, given that there is an injection $f: A \to B$.

$$A' \xrightarrow[\text{bijection}]{\theta} A \xrightarrow[\text{injection}]{f} B \xrightarrow[\text{bijection}]{\phi} B'$$

$g = \phi \circ f \circ \theta : A' \to B'$ is an injection, so $\overline{A'} \le \overline{B'}$.

$[a]+[b] = [a+b]$

$[a] = [c]$

$[b] = [d]$

$[a+b] = [c+d]$??

well-defined

THM 1) $\overline{A} \le \overline{A}$

2) If $\overline{A} \le \overline{B}$ and $\overline{B} \le \overline{C}$, then $\overline{A} \le \overline{C}$

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Suppose $\overline{A} \le \overline{B}$ and $\overline{B} \le \overline{A}$. Is it true that $\overline{A} = \overline{B}$?

Suppose $f: A \to B$ is an injection, and $g: B \to A$ is an injection. Find a bijection $h: A \to B$.

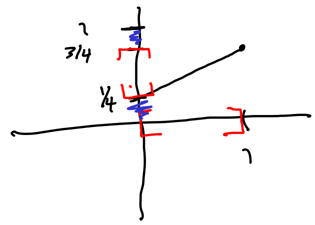
$[0,1]$, $(0,1)$

$f: (0,1) \to [0,1]$, $f(x) = x$

$g: [0,1] \to (0,1)$

$g(x) = \frac{x}{2} + \frac{1}{4}$

$\overline{[0,1]} = \overline{(0,1)}$




Bijection $h: [0,1] \to (0,1)$